A DIPOLAR CONTINUUM THEORY FOR GRANULAR MEDIA

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Corrigendum.

page

v line 13; delete " A2 Evaluation of Y_{ij} 70 ".

viii line 14; delete " a_{ij} superposed symmetric rigid body fabric velocity ".

ix line 26; make addition " $(\cdot)$ above the symbol, material time derivative, except for $\dot{e}_i$ ".

11 line 6; change to " which give the direction associated with $C_{\alpha\beta}$, and ...".

11 line 7; after the stop, insert " The vector e_i relating to the α,β direction is not necessarily in the α,β direction. It has spherical coordinate angles γ and ξ which are functions of α and β in a manner still to be defined. ".

11 line 12; change to " which have unit normals in the α,β direction. It follows...".

12 line 14; make addition " ... and ν_j is a unit vector in the α,β direction. As...".

12 lines 17, 18, 19; interchange " ν_j " and " e_i " .

13 line 1; delete the paragraph beginning " The equations (2.3) ... ".

13 line 18; delete the paragraph beginning " In a change of... ", and replace it with the following.

" From (2.3)

$$\frac{d}{dt}(e_i) = \frac{d}{dt}(E_{ij})\,\nu_j + E_{ij}\,\frac{d}{dt}(\nu_j) = \dot{E}_{ij}\nu_j + E_{ij}W_{jk}\nu_k \quad . \tag{2.7}$$

In the next chapter we will postulate expressions for the kinetic energy and the rate of working to be associated with the rate of fabric change. Anticipating this, we note that the vectors e_i primarily indicate the disposition of interparticle contacts, and we would not expect that their rotation should give rise to additional kinetic energy or rate of working in the rigid rotation of an isotropic granular body. Consequently we introduce the modified velocity term

$$\frac{d}{dt}(e_i) - E_{ij}W_{jk}\nu_k \quad . \tag{2.8}$$

For simplicity of presentation we will denote the vector (2.8) by $\dot{e}_i$, departing from our notation convention for this case only. From (2.7)

$$\dot{e}_i = \dot{E}_{ij}\,\nu_j \quad , \tag{2.9}$$

and we see that in a non isotropic granular body there will in general be a rigid rotation contribution in the $\dot{E}_{ij}$. ".

16 bottom line; change " (2.7) " to " (2.9) ".

17 line 4; delete the sentence beginning with " The components of ...", and replace it with " The components of Y_{ij} are found to be

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$$Y_{ij} = \frac{4\pi}{3}\delta_{ij} \quad ,$$

which are independent of time. ".

22 line 4; delete the term " $\dot{Y}_{ik}\dot{E}_{kj}$ " from (3.13), and renumber this equation "(3.16)".

22 line 5; delete from " We superpose a constant... " to the end of sec.3.3.

23 line 15; change to " ...help of (3.11), and the arbitrariness of V, the energy...".

65 line 15; change to " ...our assumption concerning the distribution of interparticle contacts. ".

66 line 18; delete the paragraph beginning " Finally we note... ".

67 line 4; delete the second line of equation (A.1), interchange " e_i " and " ν_j " in the rest of this sentence, and delete the equation " $J(\gamma,\xi) = \ldots$" ending it.

67 fig. A.1; interchange " $\underset{\sim}{e}$ " and " $\underset{\sim}{\nu}$ ".

67 bottom line; change to " $0 < a \leqslant b \leqslant c$ ".

68 line 1; delete from " Let $e = \|e_i\|$, ..." to the end of sec.A1, and replace it with the following.

" We now have

$$C/N = \mu\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a^2\cos^2\beta\cos^2\alpha + b^2\cos^2\beta\sin^2\alpha + c^2\sin^2\beta)^{\frac{1}{2}}\cos\beta\, d\beta\, d\alpha. \quad (A.2)$$

From Gradshteyn & Ryzhik (1965, sec.2.583), we first obtain

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (A^2\cos^2\beta + c^2\sin^2\beta)^{\frac{1}{2}}\cos\beta\, d\beta = c\left[1 + \frac{1-k^2}{2k}\ln\left(\frac{1+k}{1-k}\right)\right], \quad (A.3)$$

where $A^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$, and $0 \leqslant k^2 = 1 - \frac{A^2}{c^2} \leqslant 1$.

We note that for a possible singularity in (A.3) at $k = 0$, we have $a = b = c$ with which we find that $I = 2c$. Since we can have $k^2 = 1$ only when $a = 0$, which is not physically reasonable, we ignore this possibility. (A.1) now becomes

$$C/N = \mu c\int_0^{2\pi}\left(1 + \frac{1-k^2}{2k}\ln\left(\frac{1+k}{1-k}\right)\right)d\alpha \quad . \quad (A.4)$$

An analytic expression for (A.4) has not been found, but we assume that it could be integrated numerically if need be. In the simpler conditions of ch.6 it is

$$C/N = 2\pi\mu c\left(1 + \frac{1-k^2}{2k}\ln\left(\frac{1+k}{1-k}\right)\right) \text{ , where } k^2 = 1 - \frac{a^2}{c^2} \text{ , and } a = b.".$$

70 line 6; delete all of sec.A2.

SYNOPSIS

It is shown that the local anisotropy suggested by Horne to be related to the stress at a point in a granular medium, can be represented by a second order tensor $\underset{\sim}{E}$ at the point.

In a manner similar to that of Green and Rivlin, this tensor is then introduced into the balance equations for moment of momentum and energy, and an additional equation analogous to the balance of linear momentum but involving $\underset{\sim}{E}$ in place of the spatial position coordinate x_i .

Admissible thermodynamic processes in the medium are defined, and an initial constitutive assumption is made that the Helmholtz free energy (ψ) at a point is a function of the density (ρ) of the material, temperature (θ), temperature gradient $(\nabla\theta)$, rate of deformation $(\underset{\sim}{D})$, $\underset{\sim}{E}$, and gradient of $\underset{\sim}{E}$ $(\nabla\underset{\sim}{E})$, at the point, implying that the material is some kind of fluid.

It is shown, using the Clausius-Duhem entropy inequality and the functional analysis methods of Coleman and Mizel, that the assumed dependence of ψ upon $\nabla\theta$, $\underset{\sim}{D}$, and $\underset{\sim}{E}$ does not exist, and hence ψ is taken to be a function of ρ, θ, and $\nabla\underset{\sim}{E}$ only. It is also found that the equilibrium stress $(\underset{\sim}{T}^{(o)})$ is given, in Cartesian components, by

$$T^{(o)}_{ij} = -\rho^2 \frac{\partial\psi}{\partial\rho}\delta_{ij} - \rho\frac{\partial\psi}{\partial E_{kl,j}}E_{kl,i}$$

where δ_{ij} is the Kronecker delta, indicating that the material may sustain shear stress in equilibrium.

Using methods due to Spencer an approximate representation for ψ is constructed in terms of second degree invariants of $\nabla\underset{\sim}{E}$, and, with this, a representation for $\underset{\sim}{T}^{(o)}$ is obtained.

In a 1-dimensional stress situation, and with an internal constraint on $\underset{\sim}{E}$, a solution is obtained for the direct stress in the transverse directions, showing it to be a function of the principal direction coordinate in a second degree term and a linear term.

ACKNOWLEDGEMENTS

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LIST OF FIGURES

LIST OF COMMONLY USED SYMBOLS

A^{α}_{ij}	second order tensor in a representation for $T^{(e)}_{ij}$
B_{ij}	fabric body force per unit mass
C , $C_{\alpha\beta}$	numbers of interparticle contacts in an arbitrary volume of granular material
D_{ij}	rate of deformation
E_{ij}	fabric tensor
F , F^{*}	frames of reference
F_{iA}	deformation gradient at X_A relative to the reference configuration
H_{ijk}	fabric stress tensor
I_{α}	coefficients in representation for ψ , $(\alpha = 0 , 1 , \ldots, 5)$
J	Jacobian of transformation of variables in integration
K	kinetic energy
K_f	kinetic energy of fabric motion
K_m	kinetic energy of ordinary (non fabric) motion
K_{ij}	internal constraint tensor
L_{ij}	velocity gradient
M	mass of material in a given volume V
M_{ij}	arbitrarily chosen value for E_{ij} at a material point Z
M_{ijk}	arbitrarily chosen value for $E_{ij,k}$ at a material point Z
N	number of granules in an arbitrary volume of granular material
N_{ij}	arbitrarily chosen value for D_{ij} at a material point Z
Q	rate of change of heat energy
Q_{ij}	orthogonal transformation
R_{ij}	rotation transformation
T_{ij}	stress tensor

$T^{(e)}_{ij}$	extra stress tensor
$T^{(o)}_{ij}$	equilibrium stress tensor
U	internal energy
V	volume
W	mechanical power
W_f	mechanical power of fabric motion
W_m	mechanical power of ordinary (non fabric) motion
W_{ij}	spin tensor
X , Z	material point
X_A, Z_A	position of a material point in reference configuration
Y_{ij}	fabric inertia tensor
a , b , c	principal components of E_{ij}
a_i	superposed rigid body translational velocity
a_{ij}	superposed symmetric rigid body fabric velocity
a_{ij}	arbitrary skew symmetric tensor in superposed rigid body rotation
b_i	body force per unit mass
b_{ij}	additional to $\mathring{Y}_{ij}$ in superposed rigid body fabric motion
c_i	vector function of time in change of frame of reference
e_i	vector indicating the average number of interparticle contacts per granule, with normals in a given direction
f_i	generalised body force associated with e_i
h_{ij}	generalised stress tensor associated with e_i
ℓ	half the separation of the bounding planes in 1-dimensional boundary problem
m_i	arbitrarily chosen value for $\theta_{,i}$ at a material point Z
n_i	outward unit normal to surface of a volume V
p , q	numbers of interparticle contacts on one granule in a granular medium
p , q	principal components of the transversely isotropic equilibrium stress tensor
q_i	rate of flow of heat per unit area

r	rate of generation of heat per unit mass
t	time
x_i , z_i	spatial position of a material point
α , β	angular spherical coordinates of e_i
α	arbitrarily chosen value for ρ at a material point Z
β	arbitrarily chosen value for θ at a material point Z
γ , ξ	angular spherical coordinates of ν_i
δ_{ij}	Kronecker delta
ϵ	specific internal energy
η	specific entropy
θ	temperature
λ	undetermined multiplier (Lagrange multiplier)
μ	scalar constant in the definition of the average number of interparticle contacts per granule
μ_α	coefficients in representation for $T_{ij}^{(e)}$
ν_i	arbitrary unit vector
ρ	mass density
ρ_r	mass density in reference configuration
σ	generic term for an expression
$\varkappa_i$	deformation function
ψ	Helmholtz free energy.

OTHER NOTATION

A , B	tensor subscripts relating to the reference configuration
i , j , k,..	tensor subscripts relating to the deformed configuration
α	non-tensorial subscripts or superscripts
$(\cdot)$	above the symbol, material time derivative, except for $\dot{e}_i$
(o)	above the symbol, an objective time derivative
(')	superscript, partial derivative with respect to ρ
(*)	superscript, components evaluated in the F* frame of reference
((e))	superscript, non-equilibrium part
((o))	superscript, equilibrium part.

1 INTRODUCTION

Granular media are essentially two phase materials, consisting of solid finite sized particles in contact with each other, and a fluid occupying the void spaces between them. For our purposes the fluid will be taken to be massless; one physical prototype for our mathematical model being a dry sand.

Facets of the behaviour of granular media which have received attention include wave propagation in granular media, the response to loading of an assemblage of elastic particles when no relative motion between them occurs, flow in which the medium becomes fluidised, and quasi-static deformation in which relative motion between particles does occur. We will be concerned only with the latter. We will introduce parameters suitable to describe the rearrangement of the positions of particles relative to each other, and obtain expressions for the stress at a point in the medium in terms of these new parameters. Previous work on this problem may be roughly categorised into two classes, in which respectively:

i) the mechanical behaviour of the medium is deduced from the behaviour of a small number of particles, and,

ii) the medium is treated as a continuum.

1.1 Existing theories of granular behaviour

a The particulate approach

A theory which goes part of the way towards providing general constitutive equations for granular media has been formulated by Rowe (1962). Its main result, known as the stress-dilatancy equation, is obtained from a minimum energy ratio principle, given mathematical expression in arguments staged at the particle size scale. The theory is found to be in good agreement with the observed behaviour of sand samples in triaxial tests commonly performed in soil mechanics laboratories.

In commenting on Rowe's theory, Horne (1965, 1969) has suggested that an important factor in granular behaviour is anisotropy of packing of the granules. Using the stress-dilatancy equation, Horne shows that during deformation the pattern of interparticle contacts that forms,reflects the principal directions of stress. He gives this pattern mathematical meaning by a function which expresses the probability that a pair of contact surfaces will have normals in a given direction. Obviously, in an isotropic assembly of particles this function is independent of direction.

Horne analyses the behaviour of a sample of sand in a triaxial test. He shows that its ability to withstand increasing stress ratio[*], up to the point of peak stress ratio, is due to increasing anisotropy. In an initially dense sample the anisotropy reduces after peak stress ratio is reached, due to continued dilatation. There is a corresponding reduction in stress ratio to the value at which the sample deforms without volume change.

Oda (1972a,b) has shown that anisotropy of contacts does exist. Using a microscope equipped with a universal mechanical stage, he observed it in frozen-in-situ triaxial test samples. He has brought into use the word "fabric" as the name for the pattern of contacts, and the orientation of the granules too, if they have

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(*) The stress ratio referred to is the ratio of axial to radial direct stress.

directionality themselves. This word seems particularly appropriate to the purpose. It has a history of usage in the related science of geology, where it has been defined by Turner & Weiss (1963) as "the internal ordering of geometric or physical spatial data in an aggregate". They point out that the definition implies homogeneity in the aggregate since fabric can be determined only if there is some homogeneous pattern to express. Hence they add the definition of a "fabric domain" in an aggregate as being "any 3-dimensional portion that is statistically homogeneous on the scale of the domain". It is necessary to mention scale here as the concept of fabric in geological applications may refer to domains of any size from submicroscopic to the order of a major formation in the earth's crust. We will have use for the word later.

In other developments, Leussink & Wittke (1963) have arrived independently at a theory similar to Rowe's. Mogami (1965) has used a different approach. He has taken the common soil mechanics parameter the void ratio, and the distribution of the void ratio, to be fundamental parameters, and, in the manner of statistical mechanics theory, has constructed expressions for the probabilities of given configurations. With some assumptions, he has then derived various relationships between the components of the stress tensor. Winterkorn (1953) introduced concepts from the molecular theory of liquids in considering the engineering problem of the compaction of granular media. Oshima (1955) and Satake (1968) have introduced couple stresses in postulating the work done during the motion of individual finite sized grains, as has Mindlin (1964) in developing a continuum theory which he notes could have application to granular media. Finally, Brown & Evans (1972) investigate the application of couple-stress theories to granular media and conclude that couple-stress effects are apparently of little importance.

b The continuum approach

Many workers have applied various yield criteria in plasticity theories of the behaviour of granular media. The most common has been the Mohr-Coulomb yield criterion, an account of the use of which, under various boundary conditions, may be found in Sokolovskii (1965). Later developments have mainly been based on the work of Drucker & Prager (1952). Generally the plasticity theories have been very satisfactory in providing solutions for the stresses in 2-dimensional boundary value problems, but not as good in describing the kinematical behaviour. An important development, known as the Critical State theory, has been constructed by a group led by Roscoe at Cambridge University. It is essentially a plasticity theory incorporating porosity directly into the constitutive equations. A survey of this theory may be found in Schofield & Wroth (1968).

Recently, Goodman & Cowin (1971, 1972) have formulated a continuum theory for granular media. In addition to the usual variables of continuum mechanics, they include another kinematically independent variable, which they call the volume distribution function[*]. This is to account for void volume. The additional variable generates extra terms in the usual field equations and allows the postulation of further field equations. Thermodynamic arguments applied to the field equations, and to certain constitutive assumptions, give expressions for the stress tensor in static equilibrium, and the dissipative part of stress in non-equilibrium flow. From the first of these it is seen that the material can withstand shear in static equilibrium, and a limiting

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(*) In terms of the common soil mechanics parameter porosity, the volume distribution function = 1 - porosity.

equilibrium equation similar to the Mohr-Coulomb criterion is obtained. Using a representation for dissipative stress, some boundary value problems concerning flow are solved.

Goodman & Cowin's model has precedent in the liquid crystal theories of Ericksen (1960, 1961, 1962) and Leslie (1968), the director theories for an elastic medium with couple-stress of Toupin (1964), the multipolar theory of Green & Rivlin (1964a), and the micro-materials theories of Mindlin (1964), Eringen & Suhubi (1964) and Eringen (1964). In the liquid crystal theories, and the director theories for solids, the continuum has at each point a vector, or a set of vectors, as additional kinematical variables. However, Toupin (1964) states that his development may be readily generalised by the use of higher order tensors to describe the kinematical structure of a material point, and such tensors are explicitly included in the theories of Green & Rivlin (1964a), Eringen & Suhubi (1964) and Eringen (1964), and Mindlin (1964).

In all of the above general continuum theories the additional kinematical variables give rise to generalised surface and body forces either through postulated balance equations or in definitions of work done. From the surface forces, generalised stresses are either defined or obtained with the help of the relevant balance equations and the Cauchy tetrahedron procedure.

Green & Rivlin (1964b) note the appropriateness of their theory as a model for granular media. However, Rivlin (1968) warns that care must be taken to specify the physical meaning for the additional kinematical variables so that the physical interpretation of the theory is clear.

1.2 The present work

Horne's remarks concerning anisotropy, backed by Oda's verification of its presence, suggest that this is an important kinematic parameter that cannot be overlooked in any model of granular behaviour. In order to use the concept there are two problems to be faced:

i) to give a suitable mathematical expression to fabric anisotropy, and, having done so,

ii) to incorporate this expression in a set of tensorial constitutive equations.

To these two problems the present work is addressed. On the first matter it may be noted that Horne's "mean projected solid path" (m.p.s.p.)(*) is particularly well suited to his purpose. It is easily applied to physical arguments at particle scale, and in the axisymmetric situation of a triaxial test only two directions are significant. The m.p.s.p. does have generality in that it can be measured in any chosen direction, but this means that there are an infinite number on these parameters at a point. We overcome this difficulty, at some cost to generality, by introducing a second order tensor called the fabric tensor to describe the degree of anisotropy at a point.

We find a theory to incorporate this tensor in the dipolar continuum theory. This is formally similar to the continuum theories outlined above. The additional kinematical variable is a second order tensor - the fabric tensor in our case.

In Ch. 2 the necessary kinematical preliminaries are discussed, including the definition of the fabric tensor.

(*) Briefly the m.p.s.p. is the average projection in a given direction of the distance between interparticle contacts encountered by a path, taken entirely through the solid particles, in the general direction of the given direction.

Field equations are obtained in Ch. 3, the main result being a reduced entropy inequality. Constitutive assumptions made in Ch. 4 are substituted into the reduced entropy inequality, and more explicit forms then obtained by functional analysis methods. An approximate representation for the Helmholtz free energy is obtained in Ch. 5, and used to obtain constitutive equations from the results of Ch. 4. In Ch. 6 a 1-dimensional boundary value problem is investigated.

Cartesian tensor notation is used throughout.

2 KINEMATICS

2.1 Kinematics of non-polar motion

Let us clearly distinguish between a continuous medium and a granular medium. The word "particle" is often used in respect of both. In a granular medium it means a granule, or grain. In a continuous medium it means a material point. We will construct our granular model in a continuous medium.

We denote a material point in a continuous medium by X . We then define a set of rectangular Cartesian coordinates in which the motion of X is given by

$$x_i = \varkappa_i(X,t)$$

where $\varkappa_i$ is known as the deformation function and x_i is referred to as the spatial position of X at time t . A material coordinate system based on some reference configuration of the body is frequently used in continuum mechanics, particularly in elasticity theories, where the unstressed state is a convenient reference configuration. Since, in the main, our development does not require the use of material coordinates we do not introduce them here, but a material coordinate system will be introduced in Ch. 4 where we will briefly find it helpful.

The components of velocity at a point x_i at time t are given by the material time derivative

$$\dot{x}_i = \frac{d}{dt}(\varkappa_i(X,t))$$

where $\frac{d}{dt}$ denotes differentiation of the function $\varkappa_i$ with respect to t , X being kept fixed. The velocity gradient at the point x_i at time t is given by

$$L_{ij} = \dot{x}_{i,j} = \frac{\partial \dot{x}_i}{\partial x_j}$$

The symmetric part of L_{ij} is called the rate of deformation tensor

$$D_{ij} = \tfrac{1}{2} (\dot{x}_{i,j} + \dot{x}_{j,i})$$

while the antisymmetric part is called the spin tensor

$$W_{ij} = \tfrac{1}{2} (\dot{x}_{i,j} - \dot{x}_{j,i}).$$

2.2 Kinematics of anisotropy

a Concepts arising in a granular medium

In a granular medium let there be N particles in contact occupying a volume V. We assume that at all contact points the particle surfaces are sufficiently rounded that a tangent plane of contact can be found, as in fig. 2.1.

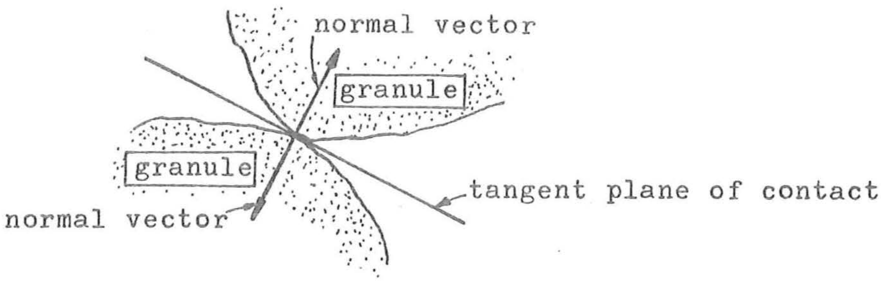

fig. 2.1

By definition we will say that there are two contacts at each contact point, as indicated by the two unit normal vectors in fig. 2.1. We will assume that the surface of V does not pass through any contact point; and if a particle cut by this surface has p contacts lying within V and q contacts in total, it will be said to contribute the fraction p/q to the number N of particles within V, and N may be not an integer. If the total number of contacts in V is C, the average number of contacts per particle is C/N .

Now we assume that V is a fabric domain in that a pattern of contacts may be discerned throughout V . The direction of a unit normal vector at a contact can be described by spherical coordinate angles α, β with respect to some fixed rectangular cartesian coordinate system, as in fig. 2.2.

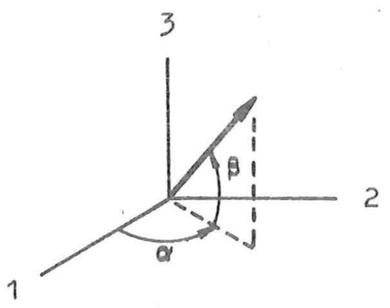

fig. 2.2

Let $C_{\alpha\beta}$ be the total number of contacts in V whose normals lie in the small solid angle $\cos\beta d\beta d\alpha$ within the limits α to $\alpha + d\alpha$, β to $\beta + d\beta$. Then we may refer to $C_{\alpha\beta}/N$ as the average number of these contacts per particle. Clearly the symmetry condition

$$C_{\alpha\beta} = C_{(\alpha+\pi)(-\beta)}$$

must apply, since at any contact point, two contact normals exist. We assume that for a sufficiently large volume (and hence sufficiently large N), $C_{\alpha\beta}$ is given by a smooth function of α, β .

b The definition of fabric in a continuous medium

We wish to introduce into a continuous medium the granular concept of the pattern of contacts. In the first part of this section we have defined C/N and $C_{\alpha\beta}/N$ as averages over a volume V . If the fabric changes sufficiently smoothly throughout the granular medium, these could be said to be values at the centroid of V . Hence we now assume that we can define these quantities at a point in a continuous medium.

We define a vector function $e_i(X,t,\alpha,\beta)$ such that at the point X at time t

$$C_{\alpha\beta}/N = \mu\sqrt{e_i e_i} \quad \cos\beta d\beta d\alpha \qquad (2.1)$$

where μ is a scalar constant, α and β are spherical coordinates which give the direction of the vector e_i, and $\cos\beta d\beta d\alpha$ is an elemental solid angle in the α, β position. The scalar μ is given dimensions $(\text{length})^{-2}$ and its value is left unspecified at present in order to simplify the definition of kinetic energy, as will be explained in sec.3.1. According to (2.1) the vector e_i represents the average number of intergranular contacts per particle which have unit normals parallel to e_i. It follows that

$$C/N = \mu \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{e_i e_i} \quad \cos\beta d\beta d\alpha \ . \qquad (2.2)$$

It can be seen from (2.1) that, unlike Horne's similarly defined function $E(\omega,\psi)$ (page 81, Horne, 1965), $e_i(X,t,\alpha,\beta)$ is not a probability density function. A function comparable with Horne's could be obtained by multiplying (2.1) by N/C.

We may now state that the motion of the particles of our continuum model for a granular medium is described by the spatial coordinates

$$x_i = \varkappa_i(X,t)$$

and the infinite set of vectors

$$e_i = e_i(X,t,\alpha,\beta)$$

for all $0 \leqslant \alpha \leqslant 2\pi$, $-\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}$.

This medium is similar to the generalised continuum of Green & Rivlin (1966), in which the configuration of a material point is described by a finite number of "generalised" coordinates q_γ, $(\gamma = 1,\ldots,\nu)$. In the examples they give, the first three of these are the spatial position coordinates, and the remainder are vectors describing the structure of the material point.

Since our vectors e_i provide the motivation for calling our continuum a model of a granular medium, we shall return to them in setting up the field equations in the next chapter. Meanwhile, in order to avoid difficulties associated with an infinite set of vectors, we assume that they may be derived from the relationship

$$e_i(X,t,\alpha,\beta) = E_{ij}(X,t)\nu_j(\alpha,\beta) \qquad (2.3)$$

where E_{ij} is a second order symmetric tensor which we call the fabric tensor, and ν_j is a unit vector. As the vector ν_j describes a unit sphere, the vector e_i maps out an ellipsoid determined by E_{ij}. Although in (2.3) we indicate that the vector ν_j is a function of α and β, it is collinear with the vector e_i only if they both lie in the principal directions of the ellipsoid. Otherwise the direction of the vector ν_j is given by $\gamma(\neq\alpha)$ and $\xi(\neq\beta)$ which are both functions of α, β, and the lengths of the principal axes of the ellipsoid. The motion of a particle of our continuum may now be described by the spatial coordinates

$$x_i = \varkappa_i(X,t)$$

and the fabric tensor

$$E_{ij} = E_{ij}(X,t) .$$

The equations (2.3) contain a loss of generality in restricting the vector e_i to map out an ellipsoid. However, we believe that this restriction is not unreasonable. If it was desired, a less restrictive relationship could be obtained in the form

$$e_i = \left(\sum_\alpha E^\alpha_{ij}\right) \nu_j \qquad (\alpha = 1,\ldots,\nu)$$

with a finite number of symmetric second order tensors E^α_{ij} .

From (2.1) and (2.2)

$$C_{\alpha\beta}/N = \mu\sqrt{E_{ij}\nu_j E_{ik}\nu_k} \quad \cos\beta d\beta d\alpha \tag{2.5}$$

and

$$C/N = \mu \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{E_{ij}\nu_j E_{ik}\nu_k} \quad \cos\beta d\beta d\alpha \ . \tag{2.6}$$

The integration in (2.6) is performed in Appendix sec.A1.

Deresiewicz (1958) notes that various investigators have found experimental evidence for a relationship between C/N and the porosity of a granular medium. Presumably the relationship would differ for media with different shaped granules, or differently graded granule sizes. In view of the present lack of knowledge on the matter we refrain from suggesting any relationship, based on (2.6), between E_{ij} and porosity at a point in a granular medium.

In a change of fabric at the material point X at time t , the ellipsoid represented by E_{ij} may both change its shape and undergo a rotation of its principal axes relative to the spatial coordinate axes. While it is clear that the vector $e_i(\alpha\ ,\ \beta)$ will indicate the average number of contacts per particle with normals in the α , β direction, it is not clear what we should take to be $\dot{e}_i$, the material time derivative of e_i . For simplicity of presentation, in view of (2.3), we now define

$$\dot{e}_i(\alpha,\beta) = \dot{E}_{ij}\nu_j(\gamma,\xi) \qquad (2.7)$$

at the point X at time t. From this definition we see that the direction of the vector e_i associated with the fixed unit vector ν_i will, in general, change as E_{ij} changes, which means that the vectors e_i and $\dot{e}_i$ will not, in general, have the same direction.

3 FIELD EQUATIONS

We begin the continuum theory with an equation expressing balance of energy. By considering a rigid body translation superposed on the motion of the continuum, we derive the balance of mass and linear momentum from the energy balance equation. Then, by analogy with the linear momentum balance, we postulate a balance for an additional quantity associated with anisotropy. The effect on the energy balance of a rigid body rotation superposed on the motion of the continuum gives us a balance of moment of momentum, and we complete the theory by postulating an entropy inequality.

Before proceeding we must clarify what we mean by mass density. To define the mass density at a point in a granular medium we must consider a mass M of material occupying a finite volume V surrounding the point. Then we define

$$\rho = \frac{M}{V}$$

to be the density at the point when V tends to some lower limit $\overset{*}{V}$, above which the alteration of V to include an additional small number of granules does not alter the ratio M/V. The value given by this definition is what we take to be the mass density at a material point in our continuum model.

3.1 Energy balance

For an arbitrary body of material of volume V, balance of energy in any deformation is given by

$$\dot{U} + \dot{K} = Q + W$$

where U is the internal energy, K is the kinetic energy, Q is the rate of change of heat energy, and W is the mechanical power in the body. K and W may be subdivided into parts K_m and W_m due to the ordinary motion of the body, and K_f and W_f, being additional, due to the fabric motion, thus

$$K = K_m + K_f \ , \quad \text{and} \quad W = W_m + W_f \ .$$

In postulating the form of K_f, Green and Rivlin (1966) would have an expression such as

$$K_f = \int_V \left(\dot{x}_i \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b\dot{e}_i \cos\beta d\beta d\alpha\right) dV$$

$$+ \int_V \left(\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{e}_i \left(\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} c\dot{e}_i \cos\xi d\xi d\gamma \right) \cos\beta d\beta d\alpha \right) dV$$

where the $b = b(\alpha,\beta)$ and $c = c(\alpha,\beta,\gamma,\xi)$ are scalar coefficients indicating differing magnitudes of mass to be associated with each scalar product inside the integrals. We can see no physical reasons to suggest that inner products between $\dot{x}_i$ and $\dot{e}_i$ or between $\dot{e}_i(\alpha_1,\beta_1)$ and $\dot{e}_i(\alpha_2,\beta_2)$ should affect the kinetic energy in a granular medium. Consequently in assuming that some kinetic energy is generated by a change in fabric irrespective of overall changes in position, we also assume that it can be given by a quadratic form in the variables $\dot{e}_i$, and we postulate that

$$K_f = \int_V \rho \left(\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{1}{2}\dot{e}_i\dot{e}_i \cos\beta d\beta d\alpha \right) dV,$$

which, in view of (2.7), becomes

$$W_f = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\oint_{\partial V} \dot{e}_j h_{jk} n_k dA + \int_V \rho\dot{e}_j f_j dV\right) \cos\beta d\beta d\alpha$$

$$= \oint_{\partial V} \dot{E}_{ji} \left(\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nu_i h_{jk} \cos\beta d\beta d\alpha\right) n_k dA$$

$$+ \int_V \rho\dot{E}_{ji} \left(\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nu_i f_j \cos\beta d\beta d\alpha\right) dV \ . \tag{3.2}$$

From an inspection of (3.2) we define a new generalised stress tensor

$$H_{ijk} = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nu_i h_{jk} \cos\beta d\beta d\alpha \tag{3.3}$$

which we call the fabric stress, and a new generalised body force per unit mass

$$B_{ij} = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nu_i f_j \cos\beta d\beta d\alpha \tag{3.4}$$

which we call the fabric body force. With these definitions we now write

$$W_f = \oint_{\partial V} \dot{E}_{ji} H_{ijk} n_k dA + \int_V \rho\dot{E}_{ji} B_{ij} dV \ . \tag{3.5}$$

Since there is no loss in generality in (3.5) if we do so, we assume that H_{ijk} and B_{ij} are symmetric in i and j .

With the help of (3.1) and (3.5), we postulate an energy balance statement as given below,

$$\frac{d}{dt}\int_V \rho\left(\epsilon + \tfrac{1}{2}\dot{x}_i\dot{x}_i + \tfrac{1}{2}Y_{jk}\dot{E}_{ij}\dot{E}_{ik}\right)dV$$

$$= \oint_{\partial V} \left(q_k + \dot{x}_i T_{ik} + \dot{E}_{ji}H_{ijk}\right)n_k dA$$

$$+ \int_V \rho\left(r + \dot{x}_i b_i + \dot{E}_{ji}B_{ij}\right)dV \qquad (3.6)$$

where ϵ is the specific internal energy, q_k is the rate of flow of heat per unit area on ∂V, r is the rate of generation of heat per unit mass in V, T_{ij} is the usual stress tensor, and b_i is the usual body force per unit mass. In taking q_k to be positive in (3.6) we are following the convention that heat energy is flowing into V when the vector q_k makes an acute angle with the outward unit normal vector n_k .

3.2 Mass and linear momentum balances

Equation (3.6) is true for all velocity fields. In particular, we will consider rigid body motions superposed onto the motions of the continuum. The resulting motions, and other kinematical variables derived from them, will be denoted by the same letter but with an asterisk superscript. They are obtained from the relationships

$$x_i^*(t) = c_i^*(t) + Q_{im}(t)\left[x_m(t) - c_m(t)\right] \qquad (3.7)$$

and $$E_{ij}^*(t) = Q_{im}(t)Q_{jn}(t)E_{mn}(t) \qquad (3.8)$$

where $c_i(t)$ and $c_i^*(t)$ are vector functions of t, and $Q_{ij}(t)$ is a proper orthogonal tensor function of t .

A superposed motion that consists only of an added translational velocity, a_i, and brings the continuum to the same configuration that it would have had in the original motion, at some time, t, is given by

$$Q_{ij}(t) = \delta_{ij},\ c_i(t) = 0,\ c_i^*(t) = a_i t .$$

We postulate that, at time t, the values of $\dot{\epsilon}$, q_i, T_{ij}, H_{ijk}, r, b_i, B_{ij}, are the same in both motions. Then, in (3.6) we may replace $\dot{x}_i$ by $\dot{x}_i + a_i$ to get

$$\frac{d}{dt}\int_V \rho\left(\epsilon + \tfrac{1}{2}(\dot{x}_i + a_i)(\dot{x}_i + a_i) + \tfrac{1}{2}Y_{ik}\dot{E}_{ji}\dot{E}_{jk}\right)dV$$

$$= \oint_{\partial V}\left(q_k + (\dot{x}_i + a_i)T_{ik} + \dot{E}_{ji}H_{ijk}\right)n_k dA$$

$$+ \int_V \rho\left(r + (\dot{x}_i + a_i)b_i + \dot{E}_{ji}B_{ij}\right)dV ,$$

and on subtracting (3.6) from this we are left with

$$\frac{d}{dt}\int_V \rho(\dot{x}_i a_i + \tfrac{1}{2}a_i a_i)dV$$

$$= \oint_{\partial V} a_i T_{ik} n_k dA + \int_V \rho a_i b_i dV .$$

Since a_i is constant we obtain

$$\frac{d}{dt}\int_V \rho(\dot{x}_i + \tfrac{1}{2}a_i)dV = \oint_{\partial V} T_{ik} n_k dA + \int_V \rho b_i dV ,$$

which, since a_i is arbitrary, decomposes into the conservation of mass expression

$$\frac{d}{dt}\int_V \rho\, dV = 0 \qquad (3.9)$$

and the linear momentum balance

$$\frac{d}{dt}\int_V \rho\, \dot{x}_i dV = \oint_{\partial V} T_{ik} n_k dA + \int_V \rho b_i dV . \qquad (3.10)$$

Now

$$\frac{d}{dt}\int_V \rho dV = \frac{\partial}{\partial t}\int_V \rho dV + \oint_{\partial V} \rho \dot{x}_i n_i dA$$

and, if $\rho \dot{x}_i$ has continuous partial derivatives with respect to x_i, the divergence theorem may be used to give

$$\oint_{\partial V} \rho \dot{x}_i n_i dA = \int_V (\rho \dot{x}_i)_{,i} dV .$$

Then (3.9) becomes

$$\int_V (\dot{\rho} + \rho \dot{x}_{i,i}) dV = 0$$

and, since V is arbitrary we have

$$\dot{\rho} + \rho \dot{x}_{i,i} = 0 \tag{3.11}$$

which is the local form of the conservation of mass.

If $T_{ik}(X,t)$ has continuous first partial derivatives over ∂V, the divergence theorem may be used in (3.10), which then, from (3.11) and the arbitrariness of V, becomes

$$\rho \ddot{x}_i = T_{ik,k} + \rho b_i \tag{3.12}$$

which is the local form of the linear momentum balance.

3.3 Fabric momentum balance

We assume that fabric motion can occur at the point X quite independently of the motion $\varkappa_i$. Consequently, by analogy with (3.10), and bearing in mind the generalised stresses and forces introduced in (3.3) and (3.4), we postulate the balance equation

$$\frac{d}{dt}\int_V \rho Y_{ik} \dot{E}_{kj} dV = \oint_{\partial V} H_{ijk} n_k dA + \int_V \rho B_{ij} dV .$$

Assuming H_{ijk} has continuous first partial derivatives with respect to x_i, we use the same reasoning as in the previous section to put this into the local form

$$\rho(\dot{Y}_{ik}\dot{E}_{kj} + Y_{ik}\ddot{E}_{kj}) = H_{ijk,k} + \rho B_{ij} . \tag{3.13}$$

We superpose a constant fabric velocity onto the fabric motion of a material point X, such that at time t, the fabric occupies the configuration it would have had in the original motion. Providing $H_{ijk,k}$ and B_{ij} are not functions of $\dot{E}_{pq}$, their values will be unchanged, and we may substitute $\dot{E}_{kj} + a_{kj}$ for $\dot{E}_{kj}$ in (3.13), where a_{ij} is a constant symmetric second order tensor. Since Y_{ik} is a function of E_{pq},

$$\dot{Y}_{ik} = \frac{\partial Y_{ik}}{\partial E_{pq}} \dot{E}_{pq}$$

and with the superposed motion this becomes

$$\frac{\partial Y_{ik}}{\partial E_{pq}} \left(\dot{E}_{pq} + a_{pq}\right) = \dot{Y}_{ik} + b_{ik}$$

where
$$b_{ik} = \frac{\partial Y_{ik}}{\partial E_{pq}} a_{pq} .$$

Substituting these changes into (3.13) we get

$$\rho\left(\dot{Y}_{ik} + b_{ik}\right)\left(\dot{E}_{kj} + a_{kj}\right) + \rho Y_{ik}\ddot{E}_{kj} = H_{ijk,k} + \rho B_{ij}$$

and subtracting (3.13) from this we are left with

$$\dot{Y}_{ik}a_{kj} + \frac{\partial Y_{ik}}{\partial E_{pq}} a_{pq}\dot{E}_{kj} + \frac{\partial Y_{ik}}{\partial E_{pq}} a_{pq}a_{kj} = 0 . \tag{3.14}$$

Since (3.14) must hold for any arbitrary choice of a_{kj} , we choose

$$a_{kj} = \dot{E}_{kj}$$

with which (3.14) reduces to

$$\dot{Y}_{ik}\dot{E}_{kj} = 0 \; . \tag{3.15}$$

The balance equation (3.13) then becomes

$$\rho Y_{ik}\ddot{E}_{kj} = H_{ijk,k} + \rho B_{ij} \; . \tag{3.16}$$

It is a straightforward matter to show that (3.16) is equivalent to an infinite set of balance equations (one for each value of the pair (α , β)) of the form

$$\frac{d}{dt}\int_V \rho \dot{e}_j dV = \oint_{\partial V} h_{jk} n_k dA + \int_V \rho f_j dV \; . \tag{3.17}$$

In view of the obvious parallel with (3.10), we could have postulated (3.17) to initiate this section, and obtained (3.16) from it, but we would not then have been able to obtain (3.15).

3.4 Moment of momentum balance

With suitable smoothness assumptions, the help of (3.11), the arbitrariness of V , and with (3.15), the energy balance (3.6) may be written in the local form

$$\rho\left(\dot{\epsilon} + \dot{x}_i\ddot{x}_i + Y_{ik}\dot{E}_{ji}\ddot{E}_{jk}\right)$$

$$= \left(q_k + \dot{x}_i T_{ik} + \dot{E}_{ji}H_{ijk}\right)_{,k} + \rho\left(r + \dot{x}_i b_i + \dot{E}_{ji}B_{ij}\right) . \tag{3.18}$$

Now, in (3.12) and (3.16) we form inner products with $\dot{x}_i$ and $\dot{E}_{ji}$ respectively and subtract these modified equations from (3.18) to obtain

$$\rho\dot{\varepsilon} = q_{k,k} + L_{ik}T_{ik} + \dot{E}_{ji,k}H_{ijk} + \rho r \ . \tag{3.19}$$

The kinematical variables in (3.19) may be decomposed thus,

$$L_{ik} = D_{ik} + W_{ik}$$

and
$$\dot{E}_{ji,k} = \overset{o}{E}_{jik} + W_{jl}E_{li,k} + W_{il}E_{jl,k}$$

where $\overset{o}{E}_{jik}$ is a time derivative of $E_{ji,k}$ defined by the above equation. Then (3.19) becomes

$$\rho\dot{\varepsilon} = q_{k,k} + D_{ik}T_{ik} + \overset{o}{E}_{jik}H_{ijk} + \rho r$$
$$+ W_{ik}(T_{ik} + E_{kj,l}H_{jil} + E_{jk,l}H_{ijl}) \ . \tag{3.20}$$

The reason for performing the above decompositions becomes apparent if we now, in the manner of sec. 3.2, superpose an arbitrary rigid body rotation on the motion of the continuum, such that the continuum occupies the same position at some time t. As before the motion is described by (3.7) and (3.8), in which, at time t,

$$Q_{ij} = \delta_{ij}, \quad \text{and} \quad \dot{Q}_{ij} = a_{ij} \tag{3.21}$$

where a_{ij} is an arbitrary constant skew symmetric tensor. It is straightforward to show from (3.7) and (3.8) that, in general,

$$\overset{*}{D}_{ij} = Q_{im}Q_{jn}D_{mn}$$

$$\overset{o*}{E}_{ijk} = Q_{im}Q_{jn}Q_{kp}\overset{o}{E}_{mnp}$$

and
$$\overset{*}{W}_{ij} = Q_{im}Q_{jn}W_{mn} + \dot{Q}_{im}Q_{jm} \ ,$$

and, for superposed rigid body rotation, the relevant expressions are obtained by substituting from (3.21). As before, we assume that the values of $\dot{e}$, $q_{k,k}$, T_{ik}, H_{ijk} and r at time t, are the same in both motions. Then, making the substitution $W_{ik} + a_{ik}$ for W_{ik} in (3.20), and subtracting (3.20), we are left with

$$a_{ik}(T_{ik} + E_{kj,l}H_{jil} + E_{jk,l}H_{ijl}) = 0 . \qquad (3.22)$$

Since a_{ik} is an arbitrary skew-symmetric tensor, the term inside the bracket must be symmetric, i.e.,

$$T_{[ik]} + E_{[kj,l}H_{ji]l} + E_{j[k,l}H_{i]jl} = 0 , \qquad (3.23)$$

where we introduce the notation

$$T_{[ik]} = \tfrac{1}{2}\left(T_{ik} - T_{ki}\right)$$

and

$$E_{[kj,l}H_{ji]l} = \tfrac{1}{2}\left(E_{kj,l}H_{jil} - E_{ij,l}H_{jkl}\right) .$$

We take (3.23) to be the local form of a moment of momentum balance.

Since W_{ik} is anti-symmetric, and in view of (3.23), (3.20) reduces to

$$\rho\dot{e} = q_{k,k} + D_{ik}T_{ik} + \overset{o}{E}_{jik}H_{ijk} + \rho r , \qquad (3.24)$$

which, for our purposes, is the final form of the energy balance.

3.5 Entropy inequality

We now introduce the temperature $\theta(X,t)$ and the specific entropy $\eta(X,t)$, and postulate the Clausius-Duhem entropy inequality

$$\frac{d}{dt}\int_V \rho\eta dV \geq \oint_{\partial V} \frac{q_k}{\theta} n_k dA + \int_V \rho \frac{r}{\theta} dV .$$

The local form of this, obtained after applying the divergence theorem, and using (3.11) and the arbitrariness of V, is

$$\rho\dot{\eta} \geq \frac{1}{\theta} q_{k,k} - \frac{1}{\theta^2} q_k \theta_{,k} + \frac{1}{\theta} \rho r . \tag{3.25}$$

We eliminate $q_{k,k}$ and r between (3.24) and (3.25) to get

$$\rho(\dot{\varepsilon} - \theta\dot{\eta}) \leq D_{ik}T_{ik} + \overset{o}{E}_{jik}H_{ijk} + \frac{1}{\theta} q_k \theta_{,k} . \tag{3.26}$$

Now introducing the Helmholtz free energy, ψ, given by

$$\psi = \varepsilon - \eta\theta ,$$

we modify (3.26) to get

$$- \rho(\dot{\psi} + \eta\dot{\theta}) + D_{ik}T_{ik} + \overset{o}{E}_{jik}H_{ijk} + \frac{1}{\theta} q_k \theta_{,k} \geq 0 . \tag{3.27}$$

This completes the framework of field equations for our continuum theory. The main results are the mass conservation equation (3.11), linear momentum balance (3.12), fabric momentum balance (3.16), moment of momentum balance (3.23), reduced energy balance (3.24), and reduced entropy inequality (3.27), to which we return in the next chapter after making some constitutive assumptions.

It may have been reasonable, as is sometimes done in continuum mechanics, to postulate balances of mass and linear momentum from the outset. In the present theory, these are found to be independent of the structure of material points, and have the same forms as for non-polar continuum mechanics. Material point structure does affect the balance of moment of momentum, however, and the form of this balance, consistent with the energy balance, is by no means obvious. For this reason we have adopted the approach given above.

4 THERMODYNAMICS

4.1 Thermodynamic processes

Following Coleman & Noll (1963) and Coleman & Mizel (1964), we now outline what we mean by a thermodynamic process. Recalling the variables introduced in the field equations in the previous chapter, we say that a thermodynamic process in a body of granular material is fully described by the following functions whose values at the material point X and time t are:

1) The spatial position $x_i = \varkappa_i(X,t)$.
2) The fabric tensor $E_{ij} = E_{ij}(X,t)$.
3) The stress tensor $T_{ij} = T_{ij}(X,t)$.
4) The body force $b_i = b_i(X,t)$ per unit mass.
5) The fabric stress tensor $H_{ijk} = H_{ijk}(X,t)$.
6) The fabric body force $B_{ij} = B_{ij}(X,t)$ per unit mass.
7) The specific internal energy $\epsilon = \epsilon(X,t)$.
8) The heat flux vector $q_i = q_i(X,t)$.
9) The heat supply $r = r(X,t)$ per unit mass.
10) The specific entropy $\eta = \eta(X,t)$.
11) The temperature $\theta = \theta(X,t)$.

We say that this set of functions is a thermodynamic process if the balances of linear momentum (3.12), fabric momentum (3.16) and energy (3.24), are satisfied, and the Clausius-Duhem inquality (3.27), is obeyed. We do not include the balance of moment of momentum here since, in the form in which we postulate it, it contains no variables not already included in the other balance equations. In non-polar continuum mechanics it is no more than a symmetry restriction on the stress tensor T_{ij}.

In the above set of functions we may distinguish three types:

1) The kinematical variables $\varkappa_i$ and E_{ij} , and the temperature θ .

2) The body forces b_i and B_{ij} , and the source of heat energy r . These are due to the action of agencies external to the material body; such as a gravitational field or an external source of radiant heat.

3) All the remaining functions T_{ij}, H_{ijk}, q_i, ϵ and η .

In order to completely describe a thermodynamic process in a body it is sufficient to prescribe the functions $\varkappa_i$, E_{ij}, θ, T_{ij}, H_{ijk}, q_i, ϵ, and η . The remaining functions b_i , B_{ij} and r are then determined by the balance equations (3.12), (3.16) and (3.24). These latter functions are singled out because they depend on external agencies and may be changed independently of the motion of the body. Changing the environment external to the body can change the values of these functions, so they can, in principle, be given values arbitrarily. Hence their inclusion in the above mentioned balance equations means that these equations do not represent constraints on the other variables appearing in them.

Note that we have defined the fabric body force B_{ij} at a material point X at time t in terms of the infinite set of body force vectors $f_i(\alpha,\beta)$ associated with the kinematical vectors $e_i(\alpha,\beta)$. The vector $f_i(\alpha_1,\beta_1)$ depends on external agencies in some unspecified way, but we assume that $f_i(\alpha_2,\beta_2)$ has essentially the same dependence, and that these two vectors are related in a manner dependent on the relationship between $e_i(\alpha_1,\beta_1)$ and $e_i(\alpha_2,\beta_2)$ contained in the tensor E_{ij} . The body force B_{ij}, incorporating the complete set of $f_i(\alpha,\beta)$, may thereby depend on E_{ij}. Notwithstanding this, sufficient arbitrariness remains in B_{ij} for the fabric momentum balance to not be a restriction on the other variables appearing in it.

4.2 Constitutive assumptions

To be able to proceed further with our continuum model, we now need to make constitutive assumptions. These will be restrictions on the thermodynamic processes, and processes which obey them will be referred to as admissible processes.

Following Truesdell & Noll (1965), we invoke four principles which all constitutive relationships must obey. These are the principles of determinism, local action, material frame indifference and equipresence.

We make constitutive assumptions by declaring that certain of the variables defining a thermodynamic process are functions of certain others. The dependent functions we call response functions and the variables they depend on are called the arguments of these functions. In choosing which variables shall be functions of what, we are guided by the principle of determinism. This states that the response of a body is determined by the history of the motion of that body. Clearly the type 1) variables of the previous section are to provide the argument variables and the type 3) variables are to be response functions. As is reasonable, since they are due to agencies external to the material body, the type 2) variables have no part in the constitutive relationships at all.

The second principle, the principle of local action, states that in determining the response functions at a given material point X, the motion outside an arbitrary neighbourhood of X may be disregarded.

This means that for argument variables we may draw on the set of kinematically independent variables and their time and space derivatives of all orders, as these can be defined within an arbitrarily small neighbourhood of X.

The principle of material objectivity, also known as material frame indifference, requires that constitutive equations be invariant under changes of frame of reference. A frame of reference cannot be given mathematical expression but a change of frame can. To within a rigid translation, it is expressed by the orthogonal time dependent second order tensor $Q_{pi}(t)$ which transforms the frame F^* into the orientation of the frame F. Objectivity requires that in a tensorial constitutive equation, both the response function and the argument variables must satisfy transformations of the following kinds:

$$\begin{aligned}
&\text{scalars;} && a^* = a \\
&\text{vectors;} && a_p^* = Q_{pi}(t)a_i \\
&\text{tensors;} && A_{pq}^* = Q_{pi}(t)Q_{qj}(t)A_{ij} \\
& && A_{pqr}^* = Q_{pi}(t)Q_{qj}(t)Q_{rk}(t)A_{ijk}
\end{aligned}$$

where the * superscripts imply that the function is evaluated in the frame F^*. Variables which obey transformations of the type shown are called objective.

Since the transformation $Q_{pi}(t)$ is time dependent, the two frames F^* and F are, in general, moving relative to each other. Consequently, some time derivatives of vectors and tensors do not transform in the above manner. For example, it is easy to show that L_{ij} transforms according the rule

$$L_{pq}^* = Q_{pi}Q_{qj}L_{ij} + \dot{Q}_{pi}Q_{qi} \qquad (4.1)$$

and, because of the occurrence of $\dot{Q}_{pi}Q_{qi}$, L_{ij} is not objective. Considering the symmetric and anti-symmetric parts of (4.1) we obtain

$$D^*_{pq} = Q_{pi}Q_{qj}D_{ij}$$

and
$$W^*_{pq} = Q_{pi}Q_{qj}W_{ij} + \dot{Q}_{pi}Q_{qi}$$

showing that D_{ij} is objective and W_{ij} is not.

On the basis of these remarks we now make our constitutive assumptions. Since the temperature θ is to be an argument function, we take the Helmholtz free energy, ψ, in place of the specific internal energy, ϵ, as one response function. Accordingly we propose that the material be characterised by the response functions

$$\psi,\ \eta,\ T_{ij},\ H_{ijk},\ q_i\ ,$$

which are all fully determined by the argument functions

$$\rho,\ D_{ij},\ E_{ij},\ E_{ij,k},\ \theta,\ \theta_{,i},$$

and these in turn are functions of X and t. In assuming that all the response functions are functions of all the argument functions we have invoked the principle of equipresence. In the next section we will show that in fact some of these assumped dependences do not exist. In passing, we observe that since the ordinary motion, $\varkappa_i$, of a material point enters the response functions through the mass density, ρ, and the rate of deformation tensor, D_{ij}, only, the material we have defined is in some sense a fluid. We have chosen not to include kinematic parameters appropriate to the deformation of a solid here because the mechanism of deformation in a granular medium more closely resembles fluid flow. It would be straightforward to include additional kinematic parameters; in which case the development would be more complicated but essentially similar to the present one.

We now find it convenient to identify the material point X with its position X_A $(A = 1,2,3)$ in some fixed reference configuration of the material, and to write

$$x_i = \varkappa_i(X_A, t) .$$

The gradient of $\varkappa_i$ with respect to X_A,

$$F_{iA} = F_{iA}(X_B, t) = \frac{\partial \varkappa_i}{\partial X_A}(X_B, t) ,$$

is called the deformation gradient at X_A relative to the reference configuration. The mass density in the reference configuration is assumed known, and is denoted by

$$\rho_r = \rho_r(X_A) .$$

It is easily shown that the mass conservation equation (3.1) can be written in the form

$$\rho \, |\det F_{iA}| = \rho_r .$$

Hence, when the functions

$$\varkappa_i(X,t), \quad E_{ij}(X,t), \quad \theta(X,t)$$

are known for all X and t, so also must ρ, $E_{ij,k}$, $\theta_{,i}$, and D_{ij} be known throughout the body. Then ϵ, η, T_{ij}, H_{ijk}, and q_i are determined using the constitutive response functions, and b_i, B_{ij} and r are determined from the balance equations (3.12), (3.16) and (3.24). Hence an admissible thermodynamic process is uniquely determined.

We can show that in any admissible thermodynamic process the argument functions given above are independent. To prove this we show that, given arbitrary values for the argument functions at some material point, Z, it is possible to construct at least one admissible thermodynamic process at that point.

We begin the proof by letting

$$\rho = \alpha(t), \qquad D_{ij} = N_{ij}(t)$$

$$E_{ij} = M_{ij}(t) \qquad E_{ij,k} = M_{ijk}(t)$$

$$\theta = \beta(t) \qquad \theta_{,i} = m_i(t)$$

be the values of the functions ρ, D_{ij}, E_{ij}, $E_{ij,k}$, θ, $\theta_{,i}$ at $X = Z$. Then, from the response functions, and $\psi = \epsilon - \eta\theta$, the values of the functions ϵ, η, T_{ij}, H_{ijk} and q_i are known. The balance equations (3.12), (3.16) and (3.24) must apply at Z for all t, but, since the values of the functions b_i, B_{ij}, and r are arbitrarily assignable, these equations do not provide any restrictions on the values we have chosen above for the argument functions.

Let $z_i = \kappa_i(Z,t)$, then a set of functions which will give rise to our arbitrary values is

$$\kappa_i(X_A,t) = z_i + A_{iA}(t) \left[X_A - Z_A\right] \tag{4.2}$$

$$E_{ij}(X_A,t) = M_{ij}(t) + M_{ijk}(t) \left[A_{kA}(t) \left(X_A - Z_A\right)\right] \tag{4.3}$$

$$\theta(X_A,t) = \beta(t) + a_i(t) \left[A_{iA}(t) \left(X_A - Z_A\right)\right] \tag{4.4}$$

where Z_A is the position of Z in the reference configuration, and $A_{iA}(t)$ is a second order invertible tensor function such that

$$\frac{1}{|\det A_{iA}(t)|}\,\rho_r = \alpha(t)$$

and $$\operatorname{symm}\left[\dot{A}_{iA}(t)A^{-1}_{jA}(t)\right] = N_{ij}(t) .$$

These latter two constraints represent, in general, 7 equations in the 18 components of $A_{iA}(t)$ and $\dot{A}_{iA}(t)$. Hence it is possible to find a tensor $A_{iA}(t)$ satisfying them for any choices of $\alpha(t)$ and $N_{ij}(t)$.

From (4.2)

$$F_{iA} = \frac{\partial \varkappa_i}{\partial X_A} = A_{iA}(t) ,$$

hence
$$\begin{aligned} D_{ij} &= \operatorname{symm}\left[L_{ij}\right] \\ &= \operatorname{symm}\left[\dot{F}_{iA}F^{-1}_{jA}\right] \\ &= \operatorname{symm}\left[\dot{A}_{iA}(t)A^{-1}_{jA}(t)\right] \\ &= N_{ij}(t) \end{aligned}$$

Also, at $X = Z$, from (4.3)

$$E_{ij} = M_{ij}(t) ,$$

and, since $$E_{ij}(x_i,t) = M_{ij}(t) + M_{ijk}(t)\left[x_k - z_k\right] ,$$

we have $$E_{ij,k} = M_{ijk}(t) .$$

Hence there is at least one set of fields

$$\varkappa_i(X,t),\quad E_{ij}(X,t),\quad \theta(X,t)$$

which will have the above arbitrarily given values for the argument variables at some point Z. This means that there is at least one admissible thermodynamic process which has the arbitrarily given values for the argument variables at the point Z.

We have shown that the argument variables may take on arbitrary values independently of each other in an admissible thermodynamic process. The same is true of their space and time derivatives. This fact is of crucial importance in the next section.

4.3 Constitutive assumptions in the entropy inequality

We have assumed that

$$\psi = \psi\,(\rho, D_{ij}, E_{ij}, E_{ij,k}, \theta, \theta_{,i})$$

and hence

$$\dot{\psi} = \frac{\partial\psi}{\partial\rho}\dot{\rho} + \frac{\partial\psi}{\partial E_{ij}}\dot{E}_{ij} + \frac{\partial\psi}{\partial E_{ij,k}}\overline{\dot{E_{ij,k}}}$$

$$+ \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\theta_{,i}}\overline{\dot{\theta_{,i}}} \qquad \frac{\partial\psi}{\partial D_{ij}}\dot{D}_{ij} \tag{4.5}$$

where $\overline{\dot{E_{ij,k}}} = \frac{d}{dt}(E_{ij,k})$, is the material time derivative of $E_{ij,k}$, etc.

From (3.11)

$$\dot{\rho} = -\rho\dot{x}_{i,i} = -\rho\delta_{ij}D_{ij}\,, \tag{4.6}$$

and it is shown in Appendix sec. A3 that

$$\overline{\dot{E_{ij,k}}} = \overset{o}{E}_{ijk} + W_{il}E_{lj,k} + W_{jl}E_{il,k}$$

$$- E_{ij,l}W_{lk} - E_{ij,l}D_{lk}\,. \tag{4.7}$$

With the help of (4.5), (4.6) and (4.7), the reduced entropy inequality (3.27) becomes, after some manipulation of indices,

$$D_{ij}(T_{ij} + \rho^2 \frac{\partial\psi}{\partial\rho}\delta_{ij} + \rho\frac{\partial\psi}{\partial E_{kl,j}} E_{kl,i}) + \overset{o}{E}_{ijk}(H_{jik} - \rho\frac{\partial\psi}{\partial E_{ij,k}})$$

$$- W_{ij}(\rho\frac{\partial\psi}{\partial E_{ik,l}} E_{jk,l} + \rho\frac{\partial\psi}{\partial E_{ki,l}} E_{kj,l} + \rho\frac{\partial\psi}{\partial E_{kl,i}} E_{kl,j})$$

$$- \dot{\theta}\,(\rho\frac{\partial\psi}{\partial\theta} + \rho\eta) - \rho\frac{\partial\psi}{\partial E_{ij}}\dot{E}_{ij} - \rho\frac{\partial\psi}{\partial\theta_{,i}}\dot{\overline{\theta_{,i}}} - \rho\frac{\partial\psi}{\partial D_{ij}}\dot{D}_{ij}$$

$$+ \frac{1}{\theta}\, q_i\theta_{,i} \geqslant 0 \qquad (4.8)$$

In the last section it was shown that for an admissible thermodynamic process the kinematical variables and their space and time derivatives may take on values independently of each other. Now, grouping the terms in one possible way, we can think of (4.8) as schematically equivalent to

$$(A) - (B_{ij})W_{ij} \geqslant 0 \qquad (4.9)$$

in which (B_{ij}) denotes the tensor

$$\left(\rho\frac{\partial\psi}{\partial E_{ik,l}} E_{jk,l} + \rho\frac{\partial\psi}{\partial E_{ki,l}} E_{kj,l} + \rho\frac{\partial\psi}{\partial E_{kl,i}} E_{kl,j}\right)$$

and (A) denotes all the terms of (4.8) apart from $(B_{ij})W_{ij}$. Since the variables in (A) and (B_{ij}) are all either response functions or kinematical variables, and none of them is a function of W_{ij}, the values taken by (A) and (B_{ij}) are independent of the values of W_{ij}. Consequently, for arbitrarily small positive of (A) it is permissible for $(B_{ij})W_{ij}$ to assume large positive values, thereby contravening the inequality (4.9). Hence (4.9) can only be true for all W_{ij} if

$$(B_{ij})W_{ij} = 0 . \tag{4.10}$$

Since W_{ij} is anti-symmetric, (4.10) will be true if (B_{ij}) is symmetric, i.e.,

$$\frac{\partial\psi}{\partial E_{[ik,l}} E_{j]k,l} + \frac{\partial\psi}{\partial E_{k[i,l}} E_{kj],l} + \frac{\partial\psi}{\partial E_{kl,[i}} E_{kl,j]} = 0 \tag{4.11}$$

With similar reasoning W_{ij} can be replaced in (4.9) by $\dot{E}_{ij}$, $\overline{\dot{\theta}_{,i}}$, $\dot{D}_{ij}$, $\dot{\theta}$, $\overset{o}{E}_{ijk}$ successively, giving the relationships

$$\frac{\partial\psi}{\partial E_{ij}} = \frac{\partial\psi}{\partial\theta_{,i}} = \frac{\partial\psi}{\partial D_{ij}} = 0 \tag{4.12}$$

and $$\eta = -\frac{\partial\psi}{\partial\theta} \tag{4.13}$$

and $$H_{jik} = \rho\frac{\partial\psi}{\partial E_{ij,k}} \tag{4.14}$$

The inequality (4.8) is thus reduced to

$$\left(T_{ij} + \rho^2\frac{\partial\psi}{\partial\rho}\delta_{ij} + \rho\frac{\partial\psi}{\partial E_{kl,j}}E_{kl,i}\right)D_{ij} + \frac{1}{\theta}q_i\theta_{,i} \geqslant 0 . \tag{4.15}$$

From (4.12) we see that the assumed dependence of ψ upon E_{ij}, $\theta_{,i}$, and D_{ij} does not exist. Hence our initial constitutive assumption is reduced to

$$\psi = \psi\left(\rho,\theta,E_{ij,k}\right) . \tag{4.16}$$

Also, from (4.13) and (4.14)

$$\eta = \eta\left(\rho,\theta,E_{ij,k}\right)$$

and $$H_{ijk} = H_{ijk}\left(\rho,\theta,E_{pq,r}\right) .$$

This last relationship shows that the fabric stress does not depend on E_{ij} in particular. From (3.3) we see that H_{ijk} incorporates the infinite set of generalised stress tensors $h_{ij}(\alpha,\beta)$ associated with the vectors $e_i(\alpha,\beta)$ at the point X at time t. Consequently, with similar reasoning to that used in sec. 4.1, we might conclude that H_{ijk} is a function of E_{ij}. We now see that this cannot be so, a fact which implies some (at present unknown) restrictions on the set of tensors $h_{ij}(\alpha,\beta)$.

4.4 Equilibrium processes

In its final form, (4.15), the entropy inequality can be written as

$$\sigma\left(D_{ij}, \theta_{,i}\right) \geqslant 0 \tag{4.17}$$

and in view of this we define an equilibrium process as one in which the 9 components

$$\left(D_{ij}, \theta_{,i}\right)$$

all vanish. This condition makes the left hand side of (4.15) a minimum, so we can say that in equilibrium

$$\frac{\partial\sigma}{\partial D_{ij}} = \frac{\partial\sigma}{\partial\theta_{,i}} = 0 . \tag{4.18}$$

Since D_{ij} is symmetric, we write

$$\sigma\left(D_{ij}, \theta_{,i}\right) = \sigma\left(\tfrac{1}{2}\left(D_{ij} + D_{ji}\right), \theta_{,i}\right)$$

before performing the differentiations in (4.18). Then, in differentiating, we may treat D_{ij} as being independent of D_{ji}, with the result

$$\tfrac{1}{2}\left[T_{ij} + T_{ji} + 2\rho^2 \frac{\partial\psi}{\partial\rho}\delta_{ij} + \rho\left(\frac{\partial\psi}{\partial E_{kl,j}} E_{kl,i} + \frac{\partial\psi}{\partial E_{kl,i}} E_{kl,j}\right)\right] = 0 \tag{4.19}$$

Introducing the notation

$$T_{(ij)} = \tfrac{1}{2}(T_{ij} + T_{ji}), \text{ etc.},$$

and referring to the equilibrium stress as $T^{(o)}_{ij}$, we have, from (4.19),

$$T^{(o)}_{(ij)} = -\rho^2 \frac{\partial\psi}{\partial\rho}\delta_{ij} - \rho\frac{\partial\psi}{\partial E_{kl,(j}} E_{kl,i)} \,. \tag{4.20}$$

Now, substituting from (4.14) in (3.23) we get

$$T_{[ij]} = -\rho\frac{\partial\psi}{\partial E_{[ik,l}} E_{j]k,l} - \rho\frac{\partial\psi}{\partial E_{k[i,l}} E_{kj],l} \,, \tag{4.21}$$

which, on substitution into (4.11), leaves

$$T_{[ij]} = -\rho\frac{\partial\psi}{\partial E_{kl,[j}} E_{kl,i]} \tag{4.22}$$

for all T_{ij}, including $T^{(o)}_{ij}$. We can combine (4.20) and (4.22) to get

$$T^{(o)}_{ij} = -\rho^2 \frac{\partial\psi}{\partial\rho}\delta_{ij} - \rho\frac{\partial\psi}{\partial E_{kl,j}} E_{kl,i} \,. \tag{4.23}$$

At this point we note the significance of our earlier assumption that H_{ijk} is symmetric in i and j. If this is not true, then, at most, (4.14) becomes

$$H_{(ji)k} = \rho\frac{\partial\psi}{\partial E_{ij,k}}$$

where $\quad H_{(ji)k} = \tfrac{1}{2}(H_{jik} + H_{ijk})\,,$

and we are unable to obtain (4.21) from (3.23) since, lacking an expression for $H_{[ji]k}$, we do not have an expression for H_{jik}. Hence we would be unable to write down (4.23).

Clearly, from (4.23),

$$T_{ij}^{(o)} = T_{ij}^{(o)}\left(\rho, \theta, E_{pq,r}\right). \tag{4.24}$$

If the $E_{pq,r}$ vanish, the equilibrium stress in our medium will be hydrostatic, as in the usual theory of fluid behaviour. Otherwise our medium can sustain shear stress in equilibrium. It appears from (4.23) that the equilibrium stress tensor need not be symmetric.

5 STRESS REPRESENTATION

5.1 A representation for the Helmholtz free energy

In the previous chapter we showed that the postulated functional dependence of ψ upon E_{ij}, $\theta_{,i}$ and D_{ij} did not exist. We now look to representation theorems to give us a more explicit relationship between ψ and its argument functions.

We begin by noting that, due to the principle of material frame indifference, ψ is a scalar invariant of its arguments ρ, θ and $E_{ij,k}$ under the full orthogonal group of transformations. This may be expressed in the equation

$$\psi\left(\rho,\theta,E_{pq,r}\right) = \psi\left(\rho,\theta,Q_{pi}Q_{qj}Q_{rk}E_{ij,k}\right) \tag{5.1}$$

where Q_{pi} is any orthogonal tensor. If in particular we set $Q_{pi} = -\delta_{pi}$ in (5.1), we get

$$\psi\left(\rho,\theta,E_{pq,r}\right) = \psi\left(\rho,\theta,-E_{pq,r}\right)$$

showing that ψ must be an even function of $E_{ij,k}$. Furthermore, we assume that ψ is a polynomial function of the components of $E_{ij,k}$, with the coefficients of the polynomial terms being functions of ρ and θ.

There is a well known theorem in the theory of invariants concerning the invariance of a scalar valued polynomial function of a second order tensor. It states that, under a given group of transformations, the scalar function is invariant if and only if it can be expressed as a polynomial function of the members of an integrity basis for the tensor. The integrity basis is a set of polynomial invariants of the tensor, in terms of whose members all other polynomial invariants of the tensor are polynomial functions. (cf. Spencer, 1971, sec. 1.2).

On the present case, where the argument is a third order tensor, there is little information. However, Spencer (1971, sec. 5.1) indicates that the above theorem will still apply, and we will consider ψ to be expressible as a polynomial function of the members of an integrity basis for $E_{ij,k}$ under the full orthogonal group of transformations. The main difficulty is to find this integrity basis. As suggested by Spencer (1972, private communication), a set of invariants may be obtained by forming tensor products

$$E_{ij,k} E_{lm,n} E_{pq,r} \dots$$

of any (even) degree, and contracting on pairs of indices all possible ways. He shows (Spencer, 1970) that under the full orthogonal group of transformations, an integrity basis for a fully symmetric third order tensor will contain terms of degree 2,4,6,8 and 10 in the components of the tensor, and we assume that similar terms will occur in an integrity basis for $E_{ij,k}$.

In order to resolve the difficulty of representation, we will assume that the components $E_{ij,k}$ have sufficiently small values that ψ may be approximately given by a polynomial in second degree invariants of $E_{ij,k}$. Then, contracting on pairs of indices in the product

$$E_{ij,k} E_{lm,n} ,$$

and using the symmetry

$$E_{ij,k} = E_{ji,k} ,$$

we arrive at the set of five invariants

$$E_{ij,k}\, E_{ij,k} \qquad\qquad E_{ij,k}\, E_{jk,i}$$

$$E_{ij,j}\, E_{ik,k} \qquad\qquad E_{ij,j}\, E_{kk,i}$$

$$E_{jj,i}\, E_{kk,i} .$$

Consequently, we postulate that

$$\psi\left(\rho,\theta,E_{pq,r}\right) = I_0 + I_1 E_{ij,k} E_{ij,k} + I_2 E_{ij,k} E_{jk,i}$$
$$+ I_3 E_{ij,j} E_{ik,k} + I_4 E_{ij,j} E_{kk,i}$$
$$+ I_5 E_{jj,i} E_{kk,i} \qquad (5.2)$$

where the coefficients I_α, $(\alpha = 0,1,\ldots,5)$, are functions of ρ and θ .

Although higher degree invariants should appear in this representation, it is felt, on physical grounds, that (5.2) is a reasonable approximation. Of course, it has the added advantage of simplicity.

5.2 Equilibrium stress

Recalling the equilibrium stress equation (4.23), we wish to form the partial derivatives $\dfrac{\partial\psi}{\partial E_{kl,j}}$. Since $E_{kl,j}$ is symmetric in k and l , we first rewrite (5.2), replacing $E_{pq,r}$ by $\frac{1}{2}(E_{pq,r} + E_{qp,r})$. Then, treating $E_{kl,j}$ as though it were independent of $E_{lk,j}$, we obtain

$$\frac{\partial\psi}{\partial E_{kl,j}} = 2I_1 E_{kl,j} + I_2(E_{jk,l} + E_{jl,k})$$

$$+ I_3 (\delta_{kj}E_{lm,m} + \delta_{lj}E_{km,m})$$

$$+ I_4 (\delta_{kl}E_{jm,m} + \tfrac{1}{2}\delta_{kj}E_{mm,l} + \tfrac{1}{2}\delta_{lj}E_{mm,k})$$

$$+ 2I_5\delta_{kl}E_{mm,j} \tag{5.3}$$

Substituting this into (4.23) we get

$$T^{(o)}_{ij} = -\rho\Big(\rho\frac{\partial\psi}{\partial\rho}\delta_{ij} + 2I_1E_{kl,j}E_{kl,i} + 2I_2E_{jl,k}E_{kl,i}$$

$$+ 2I_3E_{kl,l}E_{kj,i} + I_4(E_{jl,l}E_{kk,i} + E_{ll,k}E_{kj,i})$$

$$+ 2I_5E_{ll,j}E_{kk,i}\Big) \tag{5.4}$$

We now investigate the symmetry condition (4.11), which, because of the symmetry $E_{ij,k} = E_{ji,k}$, can be simplified to

$$2\frac{\partial\psi}{\partial E_{jl,k}}E_{il,k} + \frac{\partial\psi}{\partial E_{kl,j}}E_{kl,i} = 2\frac{\partial\psi}{\partial E_{il,k}}E_{jl,k} + \frac{\partial\psi}{\partial E_{kl,i}}E_{kl,j} \tag{5.5}$$

Then, substituting (5.3) into the left hand side of (5.5), we get the expression

$$I_1(4E_{jl,k}E_{il,k} + 2E_{kl,j}E_{kl,i}) + I_2(2E_{kl,j}E_{il,k} + 2E_{jk,l}E_{il,k}$$

$$+ 2E_{jl,k}E_{kl,i}) + I_3(2E_{kl,l}E_{ki,j} + 2E_{jl,l}E_{ik,k} + 2E_{kl,l}E_{kj,i})$$

$$+ I_4(2E_{kl,l}E_{ij,k} + E_{ll,j}E_{ik,k} + E_{ll,k}E_{ki,j} + E_{jl,l}E_{kk,i}$$

$$+ E_{ll,k}E_{kj,i}) + I_5(4E_{ll,k}E_{ij,k} + 2E_{ll,j}E_{kk,i})$$

which, by inspection, is identically symmetric. Hence our chosen representation for ψ is not inconsistent with (4.11), which therefore does not imply any new restrictions on the representation.

With the aid of (5.4) we will investigate an equilibrium boundary value problem in the next chapter.

5.3 Dissipative part of stress

Although we are mainly concerned with equilibrium, we include in this section a brief examination of the dissipative part of stress for completeness.

While we have been able to show that in equilibrium the stress does not depend on E_{ij}, $\theta_{,i}$, or D_{ij}, we have to assume that in dissipative flow our initial constitutive assumption

$$T_{ij} = T_{ij}\left(\rho, D_{pq}, E_{pq}, E_{pq,r}, \theta, \theta_{,p}\right) \tag{5.6}$$

still stands.

It is convenient to consider the stress as being the sum of the equilibrium stress $T^{(o)}_{ij}$ and the additional stress due to the non-equilibrium part of a thermodynamic process. Following Coleman & Mizel (1964) we call this additional stress $T^{(e)}_{ij}$, and write

$$T_{ij} = T^{(o)}_{ij} + T^{(e)}_{ij} \tag{5.7}$$

In view of (5.6) and (5.7) we assume that

$$T^{(e)}_{ij} = T^{(e)}_{ij}\left(\rho, D_{pq}, E_{pq}, E_{pq,r}, \theta, \theta_{,p}\right)$$

and, from (4.15), after substituting from (4.23) for $T^{(o)}_{ij}$, we have

$$T^{(e)}_{ij} D_{ij} + \frac{1}{\theta} q_i \theta_{,i} \geqslant 0 .$$

Spencer (1971, sec. 5.1) outlines a method for obtaining a representation for a tensor polynomial function of scalar, vector and tensor arguments. He shows that a second order tensor A_{ij} can be represented by the expression

$$A_{ij} = \sum_{\alpha} \mu_{\alpha} A^{\alpha}_{ij} \qquad (\alpha = 1, \ldots, \nu) \tag{5.8}$$

where the coefficients μ_{α} are scalar polynomial functions of the argument variables, and the A^{α}_{ij} are a finite number of second order tensors formed from the arguments.

A linear approximation for $T^{(e)}_{ij}$ can be obtained by requiring that it be given by an expression of the form (5.8) in which each term is of first degree in the variables D_{ij} or $\theta_{,i}$. But, considering the dependence of the stress on $\theta_{,i}$, material frame indifference requires that

$$T^{(e)}_{pq}\,(Q_{rs}\theta_{,s}) = Q_{pi}Q_{qj}T^{(e)}_{ij}(\theta_{,r}) \,,$$

and if $Q_{pi} = -\delta_{pi}$, this becomes

$$T^{(e)}_{pq}\,(-\theta_{,r}) = T^{(e)}_{pq}(\theta_{,r})$$

indicating that $T^{(e)}_{ij}$ must be an even function of $\theta_{,i}$. Hence these variables cannot be included in a linear representation, leaving $T^{(e)}_{ij}$ a homogeneous linear function of D_{ij}.

If, as a further simplification, the $E_{ij,k}$ are omitted from the tensorial part of the expression, (they may be included in the scalars μ_{α}), the representation takes the form

$$T_{ij}^{(e)} = \mu_1 D_{kk}\delta_{ij} + \mu_2 D_{kk}E_{ij} + \mu_3 D_{kk}E_{il}E_{lj}$$

$$+ \mu_4 D_{ij} + \mu_5 D_{ik}E_{kj} + \mu_6 D_{ik}E_{kl}E_{lj}$$

where, for $\alpha = 1,\ldots,6$,

$$\mu_\alpha = \mu_\alpha (\rho, E_{pq}, E_{pq,r}, \theta) .$$

When this simplification is not made, the resulting expression has many more terms, which, for simplicity of presentation, are given in Appendix sec. A4. In either form of representation, the extra stress vanishes when the rate of deformation is zero.

6 SOLUTION OF A SIMPLE EQUILIBRIUM BOUNDARY VALUE PROBLEM

6.1 Equilibrium equations

The static equilibrium equations obtained from (3.12) and (3.16) are, in the absence of body forces,

$$T^{(o)}_{ij,j} = 0 \tag{6.1}$$

and

$$H_{ijk,k} = 0 \tag{6.2}$$

Into these equations we may substitute expressions obtained from the representation for the Helmholtz free energy, ψ , in (5.2). In particular, we have already obtained a representation for $T^{(o)}_{ij}$ in (5.4). Furthermore, since

$$H_{ijk} = \rho \frac{\partial \psi}{\partial E_{ij,k}} \tag{4.14}$$

it is easily found from (5.3) that

$$\begin{aligned} H_{ijk} = \rho(2I_1 E_{ij,k} &+ I_2(E_{ki,j} + E_{kj,i}) + I_3(\delta_{ik}E_{jl,l} + \delta_{jk}E_{il,l}) \\ &+ I_4(\delta_{ij}E_{kl,l} + \tfrac{1}{2}\delta_{ik}E_{ll,j} + \tfrac{1}{2}\delta_{jk}E_{ll,i}) \\ &+ 2I_5\delta_{ij}E_{ll,k}) \end{aligned} \tag{6.3}$$

Before proceeding to substitute in (6.1) and (6.2) from (5.4) and (6.3) respectively, we make the simplifying assumption that the coefficients $I_\alpha(\alpha = 0,1,...,5)$ are not functions of position. Since the I_α are functions of ρ and θ , and since we will concern ourselves with isothermal static behaviour, this means we have assumed that ρ is not a function of position.

Consequently, from (5.4), (6.1), (6.2), and (6.3) we obtain

$$
\begin{aligned}
T^{(o)}_{ij,j} &= 0 \\
&= -\rho\Big(\rho\left(\frac{\partial\psi}{\partial\rho}\right)_{,j}\delta_{ij} \\
&\quad + 2I_1(E_{kl,jj}E_{kl,i} + E_{kl,j}E_{kl,ij}) \\
&\quad + 2I_2(E_{lj,kj}E_{kl,i} + E_{lj,k}E_{kl,ij}) \\
&\quad + 2I_3(E_{kl,lj}E_{kj,i} + E_{kl,l}E_{kj,ij}) \\
&\quad + I_4(E_{ll,kj}E_{kj,i} + E_{ll,k}E_{kj,ij} \\
&\qquad + E_{jl,lj}E_{kk,i} + E_{jl,l}E_{kk,ij}) \\
&\quad + 2I_5(E_{ll,jj}E_{kk,i} + E_{ll,j}E_{kk,ij})\Big) \qquad (6.4)
\end{aligned}
$$

and

$$
\begin{aligned}
H_{ijk,k} &= 0 \\
&= \rho\Big(2I_1E_{ij,kk} + I_2(E_{ki,jk} + E_{kj,ik}) \\
&\quad + I_3(E_{jk,ki} + E_{ik,kj}) + I_4(\delta_{ij}E_{kl,lk} + E_{kk,ij}) \\
&\quad + 2I_5\delta_{ij}E_{kk,ll}\Big) \qquad (6.5)
\end{aligned}
$$

Equations (6.4) and (6.5) give the field equations in terms of purely kinematical variables, for homogeneous density and temperature fields.

6.2 Equilibrium equations in a 1-dimensional kinematic situation

Simplified physical conditions, which may have application to the (axisymmetric) triaxial test and the consolidometer (Oedometer) test, both commonly used in soil mechanics, are obtained by requiring that E_{ij} be transversely isotropic about the x_3-axis.

Mathematically this is equivalent to requiring that

$$E_{ij} = R_{ik}E_{kl}R_{lj}$$

for all R_{ij} with components

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $0 \leqslant \theta \leqslant 2\pi$. It is straightforward to show that, as a result, E_{ij} has the components

$$\begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix} \tag{6.6}$$

where $$E_{11} = E_{22} \, . \tag{6.7}$$

Because of (6.7) it is clear that

$$E_{11,i} = E_{22,i} \quad \text{for all } i = 1,2,3.$$

Although a rather laborious procedure, it is instructive to expand (5.4) for the components of $T^{(o)}_{ij}$, incorporating the simplifications embodied in (6.6) and (6.7). The result, in Appendix sec. A5, shows that in general, although the transversely isotropic tensor E_{ij} is symmetric and has principal directions parallel to the coordinate axes, the stress tensor $T^{(o)}_{ij}$ is not symmetric, and does not have these directions as principal directions.

Now, simplifying the stress representation given in sec. A5 with the help of (6.9), shows that, in the terminology of (6.8),

$$\left.\begin{aligned} p &= \rho^2 \frac{\partial \psi}{\partial \rho} \\ \text{and} \qquad q &= p + 2\rho\left(AE^2_{11,3} + BE_{11,3}E_{33,3} + CE^2_{33,3}\right) \end{aligned}\right\} \quad (6.11)$$

in which the A, B, C are as defined above. If we denote differentiation with respect to ρ by a prime ($'$), we can expand $(6.11)_1$ using (6.10) to obtain

$$p = \rho^2\left(I_0' + A'E^2_{11,3} + B'E_{11,3}E_{33,3} + C'E^2_{33,3}\right) \quad (6.12)$$

Using this result, $(6.11)_2$ can now be written as

$$q = \rho^2 I_0' + (A\rho^2)'E^2_{11,3} + (B\rho^2)'E_{11,3}E_{33,3}$$
$$+ (C\rho^2)'E^2_{33,3} \quad (6.13)$$

Since $T^{(o)}_{ij}$ is diagonal, and its components are not functions of x_1 or x_2, the equilibrium equation (6.1) reduces to

$$T^{(o)}_{33,3} = 0$$

and from (6.13), the kinematical form of this is

$$0 = (A\rho^2)'2E_{11,33}E_{11,3} + (B\rho^2)'(E_{11,33}E_{33,3}$$
$$+ E_{11,3}E_{33,33}) + (C\rho^2)'2E_{33,33}E_{33,3} \quad (6.14)$$

Equations (6.5) may now be simplified using (6.6), (6.7) and (6.9). As a result, the equilibrium equations (6.2) reduce to the two equations

$$\left.\begin{aligned} 2AE_{11,33} + BE_{33,33} &= 0 \\ \text{and} \qquad BE_{11,33} + 2CE_{33,33} &= 0 \end{aligned}\right\} \quad (6.15)$$

With the field equations (6.14) and (6.15) we can now solve problems, given suitable boundary information.

In concluding this section we note that our assumption that ρ is independent of position is more appropriate to a consolidometer test than to a triaxial test. Of the two, the consolidometer test uses a smaller sample of material and generally subjects it to smaller strains. Consequently the consolidometer sample is more likely to have uniform density to start with, and remain close to that state. Evidence that triaxial test samples do not deform homogeneously can be found in Roscoe, Schofield & Thurairajah (1963).

6.3 A 1-dimensional boundary value problem

Consider a body of granular material bounded by two infinite planes a distance 2ℓ apart. Let the origin of a rectangular Cartesian coordinate system be placed at a point of space halfway between the planes; and the axes oriented so that the x_3-axis is normal to them. A deformation of the material occurs when these two planes move together or apart symmetrically about the plane $x_3 = 0$.

The physical condition of the material is 1-dimensional in that all functions of position are functions of x_3 only. In addition, we require the components of the fabric tensor E_{ij} to be even functions of x_3, i.e.,

$$E_{ij}(x_3) = E_{ij}(-x_3)$$

since the boundary conditions are symmetric about the plane $x_3 = 0$.

Solutions to the field equations (6.14) and (6.15) fall into two categories, depending on the determinant of coefficients in (6.15):

<u>Case i)</u> $4AC - B^2 \neq 0$.

Then from (6.15)

$$E_{11,33} = E_{33,33} = 0 \tag{6.16}$$

with which (6.14) is satisfied identically.

<u>Case ii)</u> $4AC - B^2 = 0$

indicating that the equations (6.15) are not independent of each other. Considering only $(6.15)_1$ we get

$$E_{11,33} = -\frac{B}{2A}E_{33,33} \tag{6.17}$$

which, integrated once, gives

$$E_{11,3} = -\frac{B}{2A}E_{33,3} + \text{constant.}$$

Now eliminating $E_{11,33}$ and $E_{11,3}$ from (6.14) leaves an equation of the form

$$E_{33,33}(E_{33,3} + \text{constant}) = 0$$

which has the solution

$$E_{33,33} = 0 \ , \quad \text{or} \quad E_{33,3} = \text{constant.}$$

Hence, from (6.17) we have

$$E_{11,33} = E_{33,33} = 0 . \tag{6.16}$$

We find therefore that the only solutions to (6.15) are the trivial solutions given by (6.16).

Integrating (6.16) twice we obtain

$$E_{11} = ax_3 + b$$

and
$$E_{33} = cx_3 + d$$

where a, b, c, d are constants of integration. In view of the requirement that all E_{ij} be even functions of x_3 we have

$$a = c = 0$$

and, since then $E_{11,3} = E_{33,3} = 0$, the solution reduces trivially to a hydrostatic state of stress.

An alternative solution is provided by the functions

$$\left.\begin{aligned} E_{11} &= a\,|x_3| + b \\ \text{and} \qquad E_{33} &= c\,|x_3| + d \end{aligned}\right\} \tag{6.18}$$

if we allow discontinuities in $E_{11,3}$ and $E_{33,3}$ at $x_3 = 0$. Since the terms

$$E^2_{11,3},\ E_{11,3}E_{33,3},\ E^2_{33,3}$$

will have the same sign (and magnitude) for all $-\ell \leqslant x_3 \leqslant \ell$, it is clear from (6.10), (6.12) and (6.13) that ψ and $T^{(o)}_{ij}$ will not have discontinuities at $x_3 = 0$, and will in fact have constant values over the range $-\ell \leqslant x_3 \leqslant \ell$. This is a satisfactory solution apart from the discontinuities in the partial derivatives of E_{ij} at $x_3 = 0$.

6.4 Internal constraints

On geometrical grounds there is the consideration that within the range of porosities usually found in granular media, the number of interparticle contacts per particle is bounded above and below. (For equal spheres the upper bound can be shown to be 12 (refer Deresiewicz, 1958), and, for the equilibrium of individual particles, the lower bound is 2). Also, there is some experimental evidence indicating the existence of statistical relationships between porosity (and hence ρ if the granular particles are incompressible) and the average number of contacts per particle (see earlier comment in sec. 2.2). Hence, when in sec. 6.1 we assumed ρ constant through a body, we may have constrained the average number of interparticle contacts to lie within a much smaller range. From (2.6) we have therefore placed bounds on E_{ij}, which we did not consider in obtaining the solution in the last section. In view of this we should investigate the response of the material with internal constraints.

Suppose the following kinematical constraint is imposed on the fabric tensor,

$$K_{ij}E_{ji} = \text{constant} \tag{6.19}$$

where K_{ij} is a symmetric tensor whose components are not functions of position or time. From (6.19) we have

$$K_{ij}\dot{E}_{ji} = 0 . \tag{6.20}$$

In view of (6.20), K_{ij} may be thought of as a pseudo fabric body force which does no work in an admissible deformation. Hence, if we add λK_{ij}, for any real valued λ, to the fabric body force, we obtain

$$B^*_{ij} = B_{ij} + \lambda K_{ij} . \tag{6.21}$$

Referring to Ch.3 we see that the balance of fabric momentum (3.16) becomes

$$\rho Y_{kj}\ddot{E}_{ij} = H_{kij,j} + \rho B^*_{ki} \tag{6.22}$$

but all other field equations remain unchanged. Consequently we can proceed to the same reduced entropy inequality as before.

Now we must examine the effect, if any, that the constraint (6.19) has on our constitutive assumptions. As before we expand ψ using the chain rule to obtain (4.8), which, for convenience we will write in the form

$$(A) - \rho \frac{\partial\psi}{\partial E_{ij}} \dot{E}_{ij} \geqslant 0 \tag{6.23}$$

where (A) denotes all the terms of (4.8) apart from $\rho \frac{\partial\psi}{\partial E_{ij}} \dot{E}_{ij}$. In sec. 4.3 we eliminated terms from the entropy inequality (4.8) when they contained a factor which could be chosen independently and arbitrarily, and was not an argument variable in the response functions. The components of $\dot{E}_{ij}$ are not constitutive argument variables, but, because of (6.20), they cannot all be chosen arbitrarily if K_{ij} is given. For example, if $K_{11} \neq 0$, then

$$\dot{E}_{11} = - \frac{1}{K_{11}} (K_{12}\dot{E}_{12} + K_{13}\dot{E}_{13} + \ldots + K_{33}\dot{E}_{33})$$

showing that $\dot{E}_{11}$ is determined when the other components of $\dot{E}_{ij}$ are known, though the latter may be chosen independently and arbitrarily.

To overcome this difficulty we adopt the Lagrange multiplier approach. Because of (6.20) we can augment (6.23) to obtain

$$(A) \quad -\rho\left(\frac{\partial\psi}{\partial E_{ij}} + \lambda K_{ij}\right)\dot{E}_{ij} \geqslant 0$$

which remains true for any value of λ. We choose λ so that

$$\rho\left(\frac{\partial\psi}{\partial E_{11}} + \lambda K_{11}\right) = 0 .$$

The components of E_{ij} other than E_{11} are all independent, hence for each of them

$$\rho\left(\frac{\partial\psi}{\partial E_{ij}} + \lambda K_{ij}\right)\dot{E}_{ij} = 0$$

by the reasoning employed in sec. 4.3. Therefore for all i,j = 1,2,3,

$$\left(\frac{\partial\psi}{\partial E_{ij}} + \lambda K_{ij}\right)\dot{E}_{ij} = 0 .$$

Then, because of (6.20), we have

$$\frac{\partial\psi}{\partial E_{ij}}\dot{E}_{ij} = 0 . \tag{6.24}$$

Since any $\dot{E}_{ij}$ may be non zero, (6.24) is only satisfied for all $\dot{E}_{ij}$ if

$$\frac{\partial\psi}{\partial E_{ij}} = 0 .$$

Therefore, as in sec. 4.3, ψ is not a function of E_{ij}. Consequently the previous representation for ψ still holds.

6.5 A 1-dimensional boundary value problem for a granular medium with an internal constraint

From (3.12), (6.21) and (6.22), the field equations for equilibrium in the absence of external body forces become

$$T^{(o)}_{ij,j} = 0 \tag{6.1}$$

and

$$H_{ijk,k} + \lambda K_{ij} = 0 \; . \tag{6.25}$$

We must include λK_{ij} in (6.25) because it is not due to an external agency but to an internal constraint.

We now return to the problem considered in sec. 6.3, where E_{ij} is diagonal and the boundary conditions make the problem a 1-dimensional one. From (6.5), with the simplifications given by (6.6), (6.7) and (6.9), the kinematical form of (6.25) is

$$\begin{aligned}
2AE_{11,33} + BE_{33,33} + 2\lambda K_{11} &= 0 \\
2AE_{11,33} + BE_{33,33} + 2\lambda K_{22} &= 0 \\
BE_{11,33} + 2CE_{33,33} + \lambda K_{33} &= 0 \\
\lambda K_{ij} = 0 \quad \text{for all } i \neq j \quad (i,j &= 1,2,3)
\end{aligned}$$

where A, B and C are as defined earlier. In view of these equations we confine our interest to a constraint tensor K_{ij} of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$$

There is no loss of generality in making the first two diagonal terms 1, since we have left the value of the right hand side of the constraint equation (6.19) unspecified. Note that the component k is not a function of time or place.

Equation (6.25) now reduces to the two field equations

$$\left.\begin{aligned} 2AE_{11,33} + BE_{33,33} + 2\lambda &= 0 \\ BE_{11,33} + 2CE_{33,33} + \lambda k &= 0 \end{aligned}\right\} \quad (6.26)$$

and the constraint equation (6.19) takes the form

$$2E_{11} + kE_{33} = m \qquad (6.27)$$

where m is a constant, independent of time and place.

We note that the stress equilibrium equation (6.1) still reduces to (6.14) since the constraint has no effect there. Our problem is now specified by (6.14), (6.26) and (6.27). Solutions to these four equations fall into two categories, again depending on the coefficients A, B and C.

<u>Case i):</u> $\qquad 4AC - B^2 \neq 0$.

Let $\qquad 4AC - B^2 = K_0^{-1}$

then, solving the pair of equations (6.26), we get

$$\left.\begin{aligned} E_{11,33} &= \lambda K_0(kB - 4C) \\ E_{33,33} &= 2\lambda K_0(B - kA) \end{aligned}\right\} \quad (6.28)$$

which, on integration with respect to x_3, give

$$\left.\begin{aligned} E_{11,3} &= \lambda K_0(kB - 4C)x_3 + K_1 \\ E_{33,3} &= 2\lambda K_0(B - kA)x_3 + K_2 \end{aligned}\right\} \quad (6.29)$$

and

$$\left.\begin{aligned} E_{11} &= \lambda K_0(kB - 4C)\,\frac{x_3^2}{2} + K_1x_3 + K_3 \\ E_{33} &= 2\lambda K_0(B - kA)\,\frac{x_3^2}{2} + K_2x_3 + K_4 \end{aligned}\right\} \quad (6.30)$$

where the K_1, K_2, K_3 and K_4 are constants of integration. A number of relationships between them follow.

If we require the components of E_{ij} to be even functions of x_3, (6.30) shows that

$$K_1 = K_2 = 0 . \quad (6.31)$$

Combining (6.27) and (6.30), and setting $x_3 = 0$, we obtain

$$2K_3 + kK_4 = m . \quad (6.32)$$

Finally, repeated differentiation of (6.27) with respect to x_3 gives

$$2E_{11,33} + kE_{33,33} = 0 \quad (6.33)$$

and on combining this with (6.28), we find the relationship

$$(kB - 2C) = -k(B - kA) . \quad (6.34)$$

The expressions in (6.28), (6.29) and (6.30) now simplify to

$$E_{33,33} = 2\lambda K_0(B - kA) = -\frac{2}{k}E_{11,33} \quad (6.35)$$

$$E_{33,3} = 2\lambda K_0(B - kA)x_3 = -\frac{2}{k}E_{11,3} \quad (6.36)$$

$$E_{33} = 2\lambda K_0(B - kA)\frac{x_3^2}{2} + K_4 = -\frac{2}{k}(E_{11} - \frac{m}{2}) .$$

A further relationship between the coefficients is now obtained from the equilibrium equation (6.14) by substituting from (6.35) and (6.36). This is

$$\left(A\rho^2\right)' \frac{k^2}{4} - \left(B\rho^2\right)' \frac{k}{2} + \left(C\rho^2\right)' = 0 . \tag{6.37}$$

The components of the stress tensor are now obtained from (6.12) and (6.13) using (6.36). They are

$$p = \rho^2 \left(I_0' + (A' \frac{k^2}{4} - B' \frac{k}{2} + C')\left(2\lambda K_0(B - kA)x_3\right)^2\right) \tag{6.38}$$

and $q = \rho^2 I_0' + \left((A\rho^2)' \frac{k^2}{4} - (B\rho^2)' \frac{k}{2} + (C\rho^2)'\right)\left(2\lambda K_0(B - kA)x_3\right)^2$

which, because of (6.37), becomes

$$q = \rho^2 I_0' . \tag{6.39}$$

<u>Case ii)</u>: $4AC - B^2 = 0$.

Again we begin with equations (6.26) which in this case are not independent, and consequently give rise to the relationships

$$\frac{B}{2A} = \frac{2C}{B} = \frac{k}{2} . \tag{6.40}$$

Solving $(6.26)_1$ and (6.33) we obtain, with the aid of (6.40),

$$E_{33,33} = -\frac{2\lambda}{kA} = -\frac{2}{k} E_{11,33} . \tag{6.41}$$

Integrating with respect to x_3 we find that

$$\left.\begin{aligned} E_{11,3} &= \frac{\lambda}{A} x_3 + K_1 \\ E_{33,3} &= -\frac{2\lambda}{kA} x_3 + K_2 \end{aligned}\right\} \tag{6.42}$$

and

$$\left.\begin{aligned} E_{11} &= \frac{\lambda}{A} \frac{x_3^2}{2} + K_1 x_3 + K_3 \\ E_{33} &= -\frac{2\lambda}{kA} \frac{x_3^2}{2} + K_2 x_3 + K_4 \end{aligned}\right\} \tag{6.43}$$

where the K_1, K_2, K_3 and K_4 are constants of integration, not necessarily the same as in Case i). However, for the same reasons as before, and with the same notation, (6.31) and (6.32) apply in this case also. Consequently, (6.42) and (6.43) simplify to

$$E_{33,3} = -\frac{2\lambda}{kA} x_3 = -\frac{2}{k} E_{11,3} \tag{6.44}$$

and
$$E_{33} = -\frac{2\lambda}{kA}\frac{x_3^2}{2} + K_4 = -\frac{2}{k}(E_{11} - \frac{m}{2}) \, .$$

Combining (6.41) and (6.44) with the equilibrium equation (6.14) we reacquire (6.37).

Now substituting from (6.44) into (6.12) and (6.13) we obtain the following expressions for the components of the stress tensor,

$$p = \rho^2 \left(I_0' + \left(A'\frac{k^2}{4} - B'\frac{k}{2} + C' \right) \left(\frac{2\lambda x_3}{kA}\right)^2 \right) \tag{6.45}$$

and
$$q = \rho^2 I_0' + \left((A\rho^2)'\frac{k^2}{4} - (B\rho^2)'\frac{k}{2} + (C\rho^2)' \right) \left(\frac{2\lambda x_3}{kA}\right)^2$$

which, because of (6.37), again reduces to

$$q = \rho^2 I_0' \, . \tag{6.46}$$

Comparing Case i) and Case ii) we find that in either, $T_{11}^{(o)}$ and $T_{22}^{(o)}$ are given by an expression which is quadratic in x_3. In both cases $T_{33}^{(o)}$ is the same and is independent of x_3. As a further indication of consistency, it is straightforward to show, using (6.34) and (6.40), that

$$K_0(B - kA) \rightarrow \frac{-1}{kA}$$

when
$$(4AC - B^2) \rightarrow 0$$

thereby reducing (6.38) to (6.45). We notice that we have been unable to determine λ , but we have assumed $\lambda \neq 0$ in our working since the alternative simply implies the absence of internal constraints. Finally, we cannot suggest any physical reasons why Case i) or Case ii) should have preference in a deformation of a granular medium.

The results we have obtained here do not appear to have been suggested before. From the absence of equipment suitable for the experimental verification of them, it would seem that the possibility that lateral stresses in a granular medium not subject to body forces may vary with position has not occurred to soil mechanics investigators.

As a footnote to this section we briefly investigate the special constraint $K_{ij} = \delta_{ij}$. The results can be read off directly from equations (6.38) and (6.45),

Case i): $$p = \rho^2 \left(I_0' + (\frac{A'}{4} - \frac{B'}{2} + C')\left(2\lambda K_0(B - A)x_3\right)^2\right)$$

Case ii): $$p = \rho^2 \left(I_0' + (\frac{A'}{4} - \frac{B'}{2} + C')\left(\frac{2\lambda x_3}{A}\right)^2\right).$$

Of course q is unchanged from

$$q = \rho^2 I_0' .$$

7 CONCLUSIONS

Beginning with Horne's concept of local anisotropy in a granular medium, we have constructed a mathematical model based on the principles of continuum mechanics, and from it obtained constitutive expressions in terms of kinematic variables, using them to solve a simple boundary value problem.

Two conclusions in particular arise from the above development. The first is that the gradient of E_{ij} is an appropriate kinematical quantity to measure in a laboratory investigation of the deformation of a sample of granular material. As mentioned in sec. 1.1, Oda (1972a,b) has taken some steps towards obtaining this information. Unfortunately, he was concerned only with obtaining data equivalent to the tensor E_{ij}, and not with its possible variation with position. If Oda's data were in greater detail, it would be of value in justifying (or not) our assumption that the distribution of interparticle contacts is ellipsoidal. As it is, it does suggest that the assumption is not unreasonable. It is clear that his experimental method involved very laborious procedures, but there does not seem to be any more attractive alternative technique presently available.

The second conclusion is that within a body of material with internal structure, the stress may vary with position in an unexpected way. This is apparent in our boundary value problem, even though the boundary conditions are particularly simple. Consequently, stress predictions for the interior of a body, based on the linear elasticity theory or the plasticity theories, may be considerably at variance with those based on the present approach. For this reason it would be of great value to obtain experimental verification of the stress solution we have obtained, bearing in mind that we are unable to say whether the coefficient of x_3^2 in (6.38) or (6.45) is positive or negative. It does not appear that laboratory equipment suitable for this purpose is currently in general use.

Since the mechanism of fabric change in a granular medium involves the movement of granules, it is intuitively to be expected that fabric motion will be related to the motion $\varkappa_i$, and it is a possible weakness in our model that we assume these two motions to be independent. Further work could be directed at investigating the relationship (if any) between the two motions, and then incorporating it into the continuum model. The particulate approach may be fruitful in aiding the postulation of a relationship, otherwise it may be possible to obtain experimental evidence to do the same.

Finally we note that in obtaining (3.15) from our postulated balance of fabric momentum (3.13), we assumed that H_{ijk} was not a function of $\dot{E}_{pq}$, and, consequently, we did not introduce $\dot{E}_{pq}$ into the list of argument variables in making our constitutive assumptions in sec. 4.2. Green & Rivlin (1964a, sec.16) discuss this point in regard to their development. They obtain a balance equation similar to (3.13) by applying a rigid body fabric motion to their energy balance equation, and point out that for it to hold, their internal energy function cannot be a function of $\dot{E}_{pq}$, or of E_{pq}. We avoid having to postulate this latter restriction but find that it eventually arises, in (4.12), from the analysis of the reduced entropy inequality.

APPENDIX

A1. Evaluation of C/N

$$C/N = \mu \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (E_{ij}E_{ik}\nu_j\nu_k)^{\frac{1}{2}} \cos\beta \, d\beta \, d\alpha$$

$$= \mu \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (E_{ij}E_{ik}\nu_j\nu_k)^{\frac{1}{2}} \cos\beta \, \left|J(\gamma,\xi)\right| d\xi \, d\gamma \qquad (A.1)$$

where α,β are the spherical coordinates of the vector e_i, and γ,ξ are the spherical coordinates of the vector ν_i as in fig. A.1 below, and

$$J(\gamma,\xi) = \det \begin{bmatrix} \dfrac{\partial\alpha}{\partial\gamma} & \dfrac{\partial\alpha}{\partial\xi} \\[2ex] \dfrac{\partial\beta}{\partial\gamma} & \dfrac{\partial\beta}{\partial\xi} \end{bmatrix} .$$

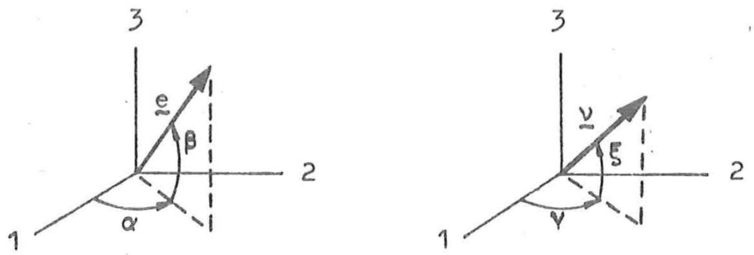

fig. A.1

There is no loss of generality if we let E_{ij} be diagonal, with components

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

where $a \geqslant b \geqslant c > 0$.

Let $e = \| e_i \|$, then the expanded form of

$$e_i = E_{ij} \nu_j$$

is

$$e \cos\beta \cos\alpha = a \cos\xi \cos\gamma \qquad (A.2)$$

$$e \cos\beta \sin\alpha = b \cos\xi \sin\gamma \qquad (A.3)$$

$$e \sin\beta = c \sin\xi \, . \qquad (A.4)$$

From (A.2) and (A.3)

$$\tan\alpha = \frac{b}{a} \tan\gamma$$

and from this we obtain

$$\frac{\partial\alpha}{\partial\gamma} = \frac{b \, (1 + \tan^2\gamma)}{a \, (1 + (\frac{b}{a})^2 \tan^2\gamma)}$$

and

$$\frac{\partial\alpha}{\partial\xi} = 0 \, .$$

From (A.4)

$$\sin\beta = \frac{c \sin\xi}{e}$$

where, in view of (A.2), (A.3) and (A.4),

$$e = \left(a^2 \cos^2\xi \cos^2\gamma + b^2 \cos^2\xi \sin^2\gamma + c^2 \sin^2\xi\right)^{\frac{1}{2}} . \qquad (A.5)$$

Therefore we get

$$\frac{\partial\beta}{\partial\gamma} = \frac{-1}{e^3 \cos\beta} \left(c(b^2 - a^2) \sin\xi \cos^2\xi \sin\gamma \cos\gamma\right)$$

and

$$\frac{\partial\beta}{\partial\xi} = \frac{c \cos\xi}{e^3 \cos\beta} (a^2 \cos^2\gamma - b^2 \sin^2\gamma) \, .$$

Hence, we obtain

$$\cos\beta \left| J(\gamma,\xi) \right| = \frac{abc \cos\xi}{e^3} . \qquad (A.6)$$

With $(E_{ij}E_{ik}\nu_j\nu_k)^{\frac{1}{2}} = e \qquad (A.7)$

and (A.6), (A.1) becomes

$$C/N = \mu abc \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\xi d\xi d\gamma}{e^2}$$

$$= \mu abc \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\xi d\xi d\gamma}{(a^2\cos^2\xi\cos^2\gamma + b^2\cos^2\xi \sin^2\gamma + c^2 \sin^2\xi)}$$

(A.8)

Changing the order of integration, we consider

$$I = 4\int_0^{\frac{\pi}{2}} \frac{d\gamma}{(A^2\cos^2\gamma + B^2\sin^2\gamma + C^2)}$$

where $A = a\cos\xi$, $B = b\cos\xi$, $C = c\sin\xi$. This integrates to give

$$I = \frac{2\pi}{\left[(A^2 + C^2)(B^2 + C^2)\right]^{\frac{1}{2}}}$$

Returning to (A.8) we now have

$$C/N = 4\pi\mu abc\int_0^{\frac{\pi}{2}} \frac{\cos\xi \, d\xi}{\left[(a^2\cos^2\xi + c^2\sin^2\xi)(b^2\cos^2\xi + c^2\sin^2\xi)\right]^{\frac{1}{2}}} .$$

With suitable changes of variable, this is found, from Gradshteyn & Ryzhik (1965, sec.3.152), to give

$$C/N = \frac{4\pi\mu ac}{\sqrt{a^2 - c^2}} F(\eta,t),$$

where $F(\eta,t)$ is an elliptic integral of the first kind, i.e.,

$$F(\eta,t) = \int_0^{\eta} \frac{dx}{\sqrt{(1 - t^2\sin^2x)}}$$

where
$$\eta = \arcsin\sqrt{\frac{a^2 - c^2}{a^2}}$$

and
$$t = \sqrt{\frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)}} .$$

A2. Evaluation of Y_{ij}

From sec.3.1

$$Y_{ij} = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nu_i\nu_j \cos\beta d\beta d\alpha$$

$$= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \nu_i\nu_j \cos\beta \left|J(\gamma,\xi)\right| d\xi d\gamma$$

where the components ν_i, in terms of the spherical coordinates defined in fig. A.1, are

$$\nu_1 = \cos\xi \cos\gamma$$

$$\nu_2 = \cos\xi \sin\gamma$$

$$\nu_3 = \sin\xi .$$

With (A.6), again considering E_{ij} to be diagonal (see sec.A1), we get

$$Y_{ij} = abc \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\nu_i \nu_j \cos\xi \, d\xi \, d\gamma}{e^3} \tag{A.9}$$

where e is given in (A.5) and the components of $\left[\nu_i \nu_j\right]$ are

$$\begin{bmatrix} \cos^2\xi \cos^2\gamma & \cos^2\xi \cos\gamma \sin\gamma & \cos\xi \sin\xi \cos\gamma \\ \cos^2\xi \cos\gamma \sin\gamma & \cos^2\xi \sin^2\gamma & \cos\xi \sin\xi \sin\gamma \\ \cos\xi \sin\xi \cos\gamma & \cos\xi \sin\xi \sin\gamma & \sin^2\xi \end{bmatrix}$$

To begin, consider the three integrals

$$I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3\xi \, d\xi}{e^3} = 2 \int_0^{\frac{\pi}{2}} \frac{\cos^3\xi \, d\xi}{e^3}$$

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\xi \sin\xi \, d\xi}{e^3} = 0$$

$$I_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\xi \sin^2\xi \, d\xi}{e^3} = 2 \int_0^{\frac{\pi}{2}} \frac{\cos\xi \sin^2\xi \, d\xi}{e^3}$$

in which γ is kept constant. We introduce

$$k^2 = 1 - \frac{c^2}{a^2\cos^2\gamma + b^2\sin^2\gamma} . \tag{A.10}$$

Recalling that in sec. A1 we took $a \geqslant b \geqslant c > 0$,

we let $y = a^2\cos^2\gamma + b^2\sin^2\gamma$

and let γ have any value in the range $0 \leqslant \gamma \leqslant \frac{\pi}{2}$. Then y will have the following values:

$$\text{at} \quad \gamma = 0, \quad y = a^2 \geqslant b^2 \geqslant c^2$$

$$\text{at} \quad \gamma = \frac{\pi}{2}, \quad y = b^2 \geqslant c^2 .$$

In the range $0 \leqslant \gamma \leqslant \frac{\pi}{2}$,

$$\frac{dy}{d\gamma} = (b^2 - a^2)\sin 2\gamma ,$$

and is always negative. Hence, over this range,

$$0 \leqslant \frac{c^2}{y} \leqslant 1$$

and it is easy to show that this is true for all γ in the range $0 \leqslant \gamma \leqslant 2\pi$. Therefore, in (A.10),

$$0 \leqslant k^2 \leqslant 1 .$$

We now have

$$\left.\begin{aligned} I_1 &= \frac{2}{(a^2\cos^2\gamma + b^2\sin^2\gamma)^{1.5}} \int_0^{\frac{\pi}{2}} \frac{\cos^3\xi d\xi}{(1 - k^2\sin^2\xi)^{1.5}} \\ I_3 &= \frac{2}{(a^2\cos^2\gamma + b^2\sin^2\gamma)^{1.5}} \int_0^{\frac{\pi}{2}} \frac{\cos\xi\sin^2\xi d\xi}{(1 - k^2\sin^2\xi)^{1.5}} \end{aligned}\right\} \quad (A.11)$$

From Gradshteyn & Ryzhik (1965, sec.2.584), these expressions integrate to give

$$I_1 = \frac{2}{(a^2\cos^2\gamma + b^2\sin^2\gamma)^{1.5}} \left[-\frac{\sqrt{1 - k^2}}{k^2} + \frac{\arcsin k}{k^3} \right]$$

$$I_3 = \frac{2}{(a^2\cos^2\gamma + b^2\sin^2\gamma)^{1.5}} \left[\frac{1}{k^2\sqrt{1 - k^2}} - \frac{\arcsin k}{k^3} \right]$$

in which it is seen that there are singularities at $k = 0$ and at $k = 1$.

Considering first $k = 0$, we find that this condition is achieved when $b = c$ and $\gamma = \frac{\pi}{2}$. These values, substituted into (A.11) give

$$\left.\begin{aligned} I_1 &= \frac{2}{c^3}\int_0^{\frac{\pi}{2}} \cos^3\xi d\xi = \frac{4}{3c^3} \\ I_3 &= \frac{2}{c^3}\int_0^{\frac{\pi}{2}} \cos\xi \sin^2\xi d\xi = \frac{2}{3c^3} . \end{aligned}\right\} \quad (A.12)$$

Alternatively, $k = 0$ when $a = c$ (implying $b = c$ also) and $\gamma = 0$, in which case (A.11) again reduce to (A.12).

Since we can have $k = 1$ only when $c = 0$, which is not reasonable on physical grounds, we ignore this possibility.

Returning to (A.9) we now have, for the six independent components of Y_{ij},

$$Y_{11} = \int_0^{2\pi} \frac{2\cos^2\gamma}{(a^2\cos^2\gamma + b^2\sin^2\gamma)^{1.5}} \left[-\frac{\sqrt{1-k^2}}{k^2} + \frac{\arcsin k}{k^3} \right] d\gamma$$

$$Y_{22} = \int_0^{2\pi} \frac{2\sin^2\gamma}{(a^2\cos^2\gamma + b^2\sin^2\gamma)^{1.5}} \left[-\frac{\sqrt{1-k^2}}{k^2} + \frac{\arcsin k}{k^3} \right] d\gamma$$

$$Y_{33} = \int_0^{2\pi} \frac{2}{(a^2\cos^2\gamma + b^2\sin^2\gamma)^{1.5}} \left[\frac{1}{k^2\sqrt{1 - k^2}} - \frac{\arcsin k}{k^3} \right] d\gamma$$

$$Y_{12} = Y_{21} = \int_0^{2\pi} \frac{2\sin\gamma\cos\gamma}{(a^2\cos^2\gamma + b^2\sin^2\gamma)^{1.5}} \left[-\frac{\sqrt{1 - k^2}}{k^2} + \frac{\arcsin k}{k^3} \right] d\gamma$$

$$Y_{23} = Y_{32} = Y_{31} = Y_{13} = 0 \qquad \text{(A.13)}$$

Although, when $b = c$, the above integrals are improper, with the integrands apparently tending to infinity as γ approaches $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, we see from (A.12) that the integrals are convergent at these points and hence can be evaluated.

The integrations in (A.13) are not simple, and analytic expressions for them have not been found. However, we have shown that they may be evaluated, and hence assume that this could be done numerically if need be.

A3. The material time derivative of $E_{ij,k}$

$$\overline{\dot{E}_{ij,k}} = \frac{\partial}{\partial t}\left(\frac{\partial E_{ij}}{\partial x_k}\right) + \frac{\partial}{\partial x_l}\left(\frac{\partial E_{ij}}{\partial x_k}\right)\dot{x}_l$$

$$= \frac{\partial}{\partial x_k}\left(\frac{\partial E_{ij}}{\partial t}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial E_{ij}}{\partial x_l}\right)\dot{x}_l$$

$$= \frac{\partial}{\partial x_k}\left(\frac{\partial E_{ij}}{\partial t}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial E_{ij}}{\partial x_l}\dot{x}_l\right) - \frac{\partial E_{ij}}{\partial x_l}\frac{\partial \dot{x}_l}{\partial x_k}$$

$$= \frac{\partial}{\partial x_k}(\dot{E}_{ij}) - E_{ij,l}\,L_{lk}$$

$$= \dot{E}_{ij,k} - E_{ij,l}\,L_{lk}\,. \tag{A.14}$$

Now, in sec. 3.4 we introduced the time derivative

$$\overset{\circ}{E}_{ijk} = \dot{E}_{ij,k} - W_{il}E_{lj,k} - W_{jl}E_{il,k}$$

with which, since $L_{lk} = D_{lk} + W_{lk}$, we get (A.14) in the form

$$\overline{\dot{E}_{ij,k}} = \overset{\circ}{E}_{ijk} + W_{il}E_{lj,k} + W_{jl}E_{il,k} - E_{ij,l}D_{lk} - E_{ij,l}W_{lk}\,. \tag{A.15}$$

A4. A representation for the extra stress $T^{(e)}_{ij}$

If

$$T^{(e)}_{ij} = T^{(e)}_{ij}\left(\rho,\ D_{pq},\ E_{pq},\ E_{pq,r},\ \theta\right)$$

is to be given a polynomial representation, homogeneous and linear in D_{pq}, then it takes the form

$$T^{(e)}_{ij} = \sum_{\alpha} \mu_{\alpha} A^{\alpha}_{ij}\ , \qquad (\alpha = 1,\ldots,\nu),$$

where $\mu_{\alpha} = \mu_{\alpha}\left(\rho,\ E_{pq},\ E_{pq,r},\ \theta\right)$

are scalar invariants under the full orthogonal group of transformations and the A^{α}_{ij} consist of the following terms:

D_{ij}	$D_{kk}\delta_{ij}$
$D_{ik}E_{kj}$	$D_{kk}E_{ij}$
$D_{ik}E_{kl}E_{lj}$	$D_{kk}E_{il}E_{lj}$
$D_{ik}E_{kj,l}E_{lm,m}$	$D_{kk}E_{ij,l}E_{lm,m}$
$D_{ik}E_{kj,l}E_{mm,l}$	$D_{kk}E_{ij,l}E_{mm,l}$
$D_{ik}E_{kl,j}E_{lm,m}$	$D_{kk}E_{il,j}E_{lm,m}$
$D_{ik}E_{kl,j}E_{mm,l}$	$D_{kk}E_{il,j}E_{mm,l}$
$D_{ik}E_{lj,k}E_{lm,m}$	$D_{kk}E_{jl,i}E_{lm,m}$
$D_{ik}E_{lj,k}E_{mm,l}$	$D_{kk}E_{jl,i}E_{mm,l}$
$D_{ik}E_{kl}E_{lj,m}E_{mn,n}$	$D_{kk}E_{il}E_{lj,m}E_{mn,n}$
$D_{ik}E_{kl}E_{lj,m}E_{nn,m}$	$D_{kk}E_{il}E_{lj,m}E_{nn,m}$

$D_{ik}E_{kl}E_{lm,j}E_{mn,n}$ $D_{kk}E_{il}E_{lm,j}E_{mn,n}$

$D_{ik}E_{kl}E_{lm,j}E_{nn,m}$ $D_{kk}E_{il}E_{lm,j}E_{nn,m}$

$D_{ik}E_{kl}E_{mj,l}E_{mn,n}$ $D_{kk}E_{il}E_{mj,l}E_{mn,n}$

$D_{ik}E_{kl}E_{mj,l}E_{nn,m}$ $D_{kk}E_{il}E_{mj,l}E_{nn,m}$

$D_{ik}E_{kl}E_{lm}E_{mj,n}E_{np,p}$ $D_{kk}E_{il}E_{lm}E_{mj,n}E_{np,p}$

$D_{ik}E_{kl}E_{lm}E_{mj,n}E_{pp,n}$ $D_{kk}E_{il}E_{lm}E_{mj,n}E_{pp,n}$

$D_{ik}E_{kl}E_{lm}E_{mn,j}E_{np,p}$ $D_{kk}E_{il}E_{lm}E_{mn,j}E_{np,p}$

$D_{ik}E_{kl}E_{lm}E_{mn,j}E_{pp,n}$ $D_{kk}E_{il}E_{lm}E_{mn,j}E_{pp,n}$

$D_{ik}E_{kl}E_{lm}E_{nj,m}E_{np,p}$ $D_{kk}E_{il}E_{lm}E_{nj,m}E_{np,p}$

$D_{ik}E_{kl}E_{lm}E_{nj,m}E_{pp,n}$ $D_{kk}E_{il}E_{lm}E_{nj,m}E_{pp,n}$

The above terms have been deduced with the aid of Table V in Spencer (1971). They are not all the possible irreducible invariants in that no terms of greater than second degree in $E_{ij,k}$ are included. This is done for the same reason as is given in sec. 5.1 above.

A5. Components of $T_{ij}^{(o)}$ with transversely isotropic E_{ij}

$$
\begin{aligned}
T_{11}^{(o)} = -\rho \Big(\rho \frac{\partial\psi}{\partial\rho}
&+ E_{11,1}^2 (4I_1 + 2I_2 + 2I_3 + 4I_4 + 8I_5) \\
&+ E_{11,1}E_{33,1}(2I_4 + 8I_5) \\
&+ E_{33,1}^2(2I_1 + 2I_5)\Big)
\end{aligned}
$$

$$
\begin{aligned}
T_{22}^{(o)} = -\rho \Big(\rho \frac{\partial\psi}{\partial\rho}
&+ E_{11,2}^2 (4I_1 + 2I_2 + 2I_3 + 4I_4 + 8I_5) \\
&+ E_{11,2}E_{33,2}(2I_4 + 8I_5) \\
&+ E_{33,2}^2(2I_1 + 2I_5)\Big)
\end{aligned}
$$

$$
\begin{aligned}
T_{33}^{(o)} = -\rho \Big(\rho \frac{\partial\psi}{\partial\rho}
&+ E_{11,3}^2 (4I_1 + 8I_5) \\
&+ E_{11,3}E_{33,3}(4I_4 + 8I_5) \\
&+ E_{33,3}^2(2I_1 + 2I_2 + 2I_3 + 2I_4 + 2I_5)\Big)
\end{aligned}
$$

$$
\begin{aligned}
T_{12}^{(o)} = T_{21}^{(o)} = -\rho \Big(&E_{11,1}E_{11,2}(4I_1 + 2I_2 + 2I_3 + 4I_4 + 8I_5) \\
&+ E_{11,1}E_{33,2}(I_4 + 4I_5) \\
&+ E_{11,2}E_{33,1}(I_4 + 4I_5) \\
&+ E_{33,1}E_{33,2}(2I_1 + 2I_5)\Big)
\end{aligned}
$$

$$T_{23}^{(o)} = -\rho\Big(E_{11,2}E_{11,3}(4I_1 + 8I_5)$$

$$+ E_{11,2}E_{33,3}(2I_4 + 4I_5)$$

$$+ E_{11,3}E_{33,2}(2I_4 + 4I_5)$$

$$+ E_{33,3}E_{33,2}(2I_1 + 2I_2 + 2I_3 + 2I_4 + 2I_5)\Big)$$

$$T_{32}^{(o)} = -\rho\Big(E_{11,2}E_{11,3}(4I_1 + 2I_2 + 2I_3 + 4I_4 + 8I_5)$$

$$+ E_{11,2}E_{33,3}(I_4 + 4I_5)$$

$$+ E_{11,3}E_{33,2}(I_4 + 4I_5)$$

$$+ E_{33,2}E_{33,3}(2I_1 + 2I_5)\Big)$$

$$T_{31}^{(o)} = -\rho\Big(E_{11.1}E_{11,3}(4I_1 + 2I_2 + 2I_3 + 4I_4 + 8I_5)$$

$$+ E_{11,1}E_{33,3}(I_4 + 4I_5)$$

$$+ E_{11,3}E_{33,1}(I_4 + 4I_5)$$

$$+ E_{33,1}E_{33,3}(2I_1 + 2I_5)\Big)$$

$$T_{13}^{(o)} = -\rho\Big(E_{11,1}E_{11,3}(4I_1 + 8I_5)$$

$$+ E_{11,1}E_{33,3}(2I_4 + 4I_5)$$

$$+ E_{11,3}E_{33,1}(2I_4 + 4I_5)$$

$$+ E_{33,1}E_{33,3}(2I_1 + 2I_2 + 2I_3 + 2I_4 + 2I_5)\Big)$$

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